Observations on Variables in Logic and Semantics

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1 Variables and other objects of wonder

1.1 Variables

> The variable *quâ* variable, the variable *an und für sich* and *par excellence*, is the bindable objectual variable. It is the essence of ontological idiom, the essence of the referential idiom.
>
> [Quine(1975), 155], to be continued in next section

(1) *Arithmetic*

a. $(a+b)^2 = a^2 + 2ab + b^2$ binomial equation

b. ... for any [...] numbers a and b.

(2) *Geometry*

a. $a^2 + b^2 = c^2$ Pythegorean theorem

b. ... where a and b are [the lengths of] the legs of a right triangle and c is [the length of] its hypothenuse.

c. Let a and b are [the lengths of] the legs of a right triangle and let c be [the length of] its hypothenuse. Then (2a) holds.

(3) *Logic*

a. $(\forall x)[P(x) \to Q(x]$ first-order predicate logic

b. ... where x is an individual understood

c. $(\forall x^e)[P(x) \to Q(x]$ functional type logic

d. $\forall(\lambda x^e.[P(x) \to Q(x])$ decomposing quantifier prefix

e. $(\lambda P^{et}.(\lambda Q^{et}.(\forall x^e)[P(x) \to Q(x]))$ ... binding predicate variables

(4) *Politics*

a. Let us call him Ahmed ... or Mamadou ... It could be Angeline, as well ...

b. They have outstanding voices, they are well-known and respected in their countries. The organiser of a festival in Bretagne ... or in Lorraine ... has invited her/him for a spring festival on this side of the globe.

c. Everything is ready: all administrative documents have been provided. But Mamadou (or Angeline) will not come.
https://on-the-move.org/about/our-news/
right-mobility-artists-please-sign-schengen-opera-petition

(5) The hallmark of the variable is its *introduction* (usually in a *prefix*, but sometimes with hindsight) specifying:

a. their *form*, based on their arbitrariness, resulting in alphabetic variation;
b. their *range*, reflecting their 'objectuality' – ignoring substitutional binding (mostly for convenience);
c. their *role*, usually expressed in terms of (or in tandem with) binding.

Note: (4a.) should only introduce one variable and be clearly separated from (4b.) and (4c.); simultaneous introductions of more than one variable needs to be disambiguated:

- $(\lambda wt.\ \varphi)$ pair abstraction
- $\equiv \ldots(\lambda w.\ (\lambda t.\ \varphi))$ vs.
- $\equiv \ldots(\lambda p.\ (\exists w)(\exists t)\ [p = (w,t) \wedge \varphi])$
- $\{f(x)|\ \varphi\}$ functional terms in set-abstractions
- $\equiv \ldots\{y|\ (\exists x)\ [y = f(x) \wedge \varphi]\}$ vs. as in: $rge(f) = \{f(x)|\ x \in dom(f)\}$
- $\equiv \ldots\{y|\ (\exists f)\ [y = f(x) \wedge \varphi]\}$ as in: $Sat_x(\varphi) = \{g(x)|\ [\![\varphi]\!]^g = 1\}$

1.2 Other (?) objects of wonder

> But it takes some distilling, for it has strong affinities with quite a variety of closely associated notions and devices.
>
> [Quine(1975), 155], continued from previous section

(6) $(\forall x)[\underline{P}(x) \rightarrow \underline{Q}(x]$ predicates in first-order logic [random logic text]

(7) If $\underline{\varphi}, \underline{\psi} \in \mathit{Wff}$, then $[\underline{\varphi} \wedge \underline{\psi}] \in \mathit{Wff}$. schematic letters

(8) If $\underline{x}$ is a variable and φ is a wff, then $(\forall \underline{x})\varphi$ is a wff. meta-variables

(9) $\underline{\mathit{Sentence}} \rightarrow \underline{NP} + \underline{VP}$ non-terminal symbols [Chomsky(1957), 26]

(10) a. No plant lost its leaves. 3rd person pronoun
b. We hook up the engine to the boxcar and send it to Corning.
ambiguous pronoun [Poesio(2021), 56]

(11) a. A bishop blesses a bishop. Indefinites
b. $\overline{[x, y : B(x), B(y)]}$ DRT representation: $x \neq y$ [Novelty]

Note: Distinctness in variables does not imply disjoint reference.
Exception: [Hintikka(1956)]

(12) *Why none of the above qualify as (regular) variables:*

a. Predicates (in first-order logic) cannot be bound.
b. Schematic letters and meta-variables are not (really) objectual; but they are variables ranging over linguistic objects (strings of symbols);
c. Non-terminals are pure strings without denotation; but they are also used as variables (and modified) over strings they dominate in a syntactic tree;
d. Pronouns cannot be introduced, but they may contain variables.
e. If anything, indefinites *introduce* variables.

2 Variables and ontology

> To be is, purely and simply, to be the value of a variable.
>
> [Quine(1948), 31]

(13) a. Every man loves a woman.

b. $(\forall x)[\mathrm{M}(x) \to (\exists y)[\mathrm{W}(y) \wedge L(x,y)]]$ predicate logic
... committed to individuals

c. $(\forall x : \mathrm{M}(x))(\exists y : \mathrm{W}(y))L(x,y)]]$ enhanced predicate logic
... committed to men and women (?)

d. $\underline{\underline{\mathrm{L}}}(\lambda \underline{P}^{et}.\ (\forall \underline{x}^{e})[\mathrm{M}(x) \to \underline{P}(x)])(\lambda \underline{P}^{et}.\ (\exists \underline{x}^{e})[\mathrm{W}(x) \wedge \underline{P}(\underline{x})]))$ [Montague(1970)]
... committed to individuals and (characteristic functions of) sets of individuals [Liefke(2024)]
... though by Quine's Criterion, not to possible worlds or (characteristic functions of) relations between quantifiers
to be revised

> It seems to me that this is the strategy employed by Montague Grammarians, who are in fact strongly committed to compositionality. [...]. There is a price to be paid however. The higher order entities evoked in this ype theoretical ascentre much less realistic philosophically and psycholinguistically than our original individuals. Hence the ascent is bound to detract from the psycholinguistic and methodological realism of one theory.
>
> [Hintikka(1983), 20]

(14) *Fregean semantics* cf. [Frege(1891)], [Frege(1892)]
The *extension* of an expression X is:

a. the referent of X if X is a term;
b. the truth value of X if X is a sentence;
c. X's *contribution* to the extensions of the terms and sentences in which it occurs, preferably to be determined by the following heuristics (fingers crossed):

d. $[\![X]\!] = \begin{cases} \text{that function } f \text{ such that, for any possible sister node } Y\text{:} \\ f([\![Y]\!]) = [\![X\ Y]\!], \text{ if such a function } f \text{ exists;} \\ \text{that function } g \text{ such that, for any possible sister node } Y\text{:} \\ g([\![Y]\!]^\wedge) = [\![X\ Y]\!], \text{ otherwise.} \end{cases}$

Note: With its abstract notion of extensional contributions, Frege's heuristics is obviously committed to a hierarchy of functions – but, as long as the truth conditions of its sentences are free of higher-order abstractions, the language analyzed by it is not.

Analogy: numbers and functions in physics vs. in the physical world

3 Free vs. bound variables

3.1 Free variables

(15) *Are there any?*

a. No: variables are introduced and thereby bound.

b. Yes: they are free in the remainder of formula (aka *matrix*).

(16) Term interpretation

a. If t is a term and P is a predicate, then (the formula) $P(t)$ is true iff the referent of t has the property attributable by P.

schematic letters featuring in interpretation

b. If t is a term and P is a predicate, then $[\![P(t)]\!]$ is true iff the referent of $[\![t]\!] \in [\![P]\!]$.

c. What about the occurrences of variables in their introduction?

(17) What is $[\![t]\!]$ if t is a variable? 3 answers (excluding substitutional interpretation):

a. Give up compositionality

b. Revise the syntax of binding.

c. Whatever it takes: the contributions of free variables are compositional artifacts. [Hodges(2001)]

d. Referents supplied by context. [Montague(1970)]

e. Arguments of relations expressed by open formulae.

[Quine(1960)]: 'predications'

(18) Interpreting variable introductions, depending on the answer to (17)

a. One option: substitutional interpretation

b. Rule out empty binding and more. [Wehmeier(2018)]

c. Introduce variable assignments for 'local' denotations and declare the ('global') meaning of a variable x a function assigning to each assignment g the value g assigns to x: $[\![x]\!] = \lambda g.g(x)$. [Zimmermann and Sternefeld(2013), ch. 10]: whiff of representationalism

d. Stretch the notion of a possible context so as to include arbitrary variable assignments; then proceed as in (18c). monstrous binding: [Rabern(2013)]

e. Variables have no (independent) meaning, which is why they can be explained away – after binding.

3.2 Explaining bound variables away

(19) a. $(\exists x)[P(x) \wedge Q(x)]$ predicate logic

b. $ext_P \cap ext_Q \neq \varnothing$ set theory

c. $\mathbf{E}\ \mathbf{R}\ \mathbf{K}PQ$ Predicate Functor Logic (*PFL*)

Note: All closed formulae of first- [and second-] order predicate logic can be expressed in [a suitable extension of] *PFL*.
See (and read) [Quine(1960)]; [Bernays(1959)] [+ [Došen(1988)]]

(20) a. $(\exists w)[w_0 R w \wedge p(w)]$ counterpart theory: [Lewis(1968)]

b. $\Diamond p$ modal propositional logic

Notes: According to the so-called standard translation from modal to predicate logic, parameters of interpretation come out as free variables and modal operators as quantifiers binding them.
The translation only covers an expressively limited fragment of the latter, excluding, e.g., $(\forall w_0)R(w_0, w_0)$. [Goranko and Otto(2007)]

(21) [Hintikka(1969)] semantics

a. $(\lambda p^{st}.\ (\lambda x.\ (\forall w^s)[\mathrm{Epi}(w_0)(x) \rightarrow p(j)]))$
... in *Ty2*[Gallin(1975)],

b. $(\lambda p^{st}.\ (\lambda x.\ (\lambda q^{st}.\ \Box\,[^\vee p \rightarrow^\vee q])(\mathrm{Epi}(x))))$
... *IL* [Montague(1970)],

c. $(\lambda p^{st}.\ (\lambda x.\ (\lambda q^{st}.\ (\forall \underline{w}_0)[p(\underline{w}_0) \rightarrow q(\underline{w}_0)])(\mathrm{Epi}(w_0)(x))))$
... and back: conversion requires renaming of bound $\underline{w}_0$

Notes: The standard translation of *IL* into *Ty2* ([Gallin(1975)]) covers almost all of the latter, missing only formulae of types outside *IL*, or containing free variables and constants of such types (like $(ss), (es), ((es)t)$, etc.). [Zimmermann(1989)]
The dramatic difference in expressive power does not carry over

to type logic, essentially because nestings of world variables can be mimicked by higher-order operators, as illustrated in (21b). Since the translation (almost) reverses, the question arises whether there is any substantive difference between variables and parameters of evaluation. The following puzzle relates to this question

(22) Anti-Propositionalism vs. Perspectivism

a. Anti-Propositionalism (special case) [Grzankowski(2013)]
Some attitudes are irreducibly attitudes towards properties.

b. Perspectivism [Lewis(1979)]
Some attitudes are irreducibly attitudes towards properties. [ignoring Hintikka reduction]

c. Question: What's the difference?

d. Answer: The perspectivist's properties are obtained by abstracting from the speaker (or epistemic center) parameter, roughly, turning the meaning of *I am hungry* into the property of being hungry; the anti-propositionalist's properties do not require such a source. (22a) and (22b) only coincide if any property can be thought of as being derived from a subjective proposition.

e. John likes chocolate. [Montague(2007)]

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