Remarks on the non-standard interpretation of intensional type logic

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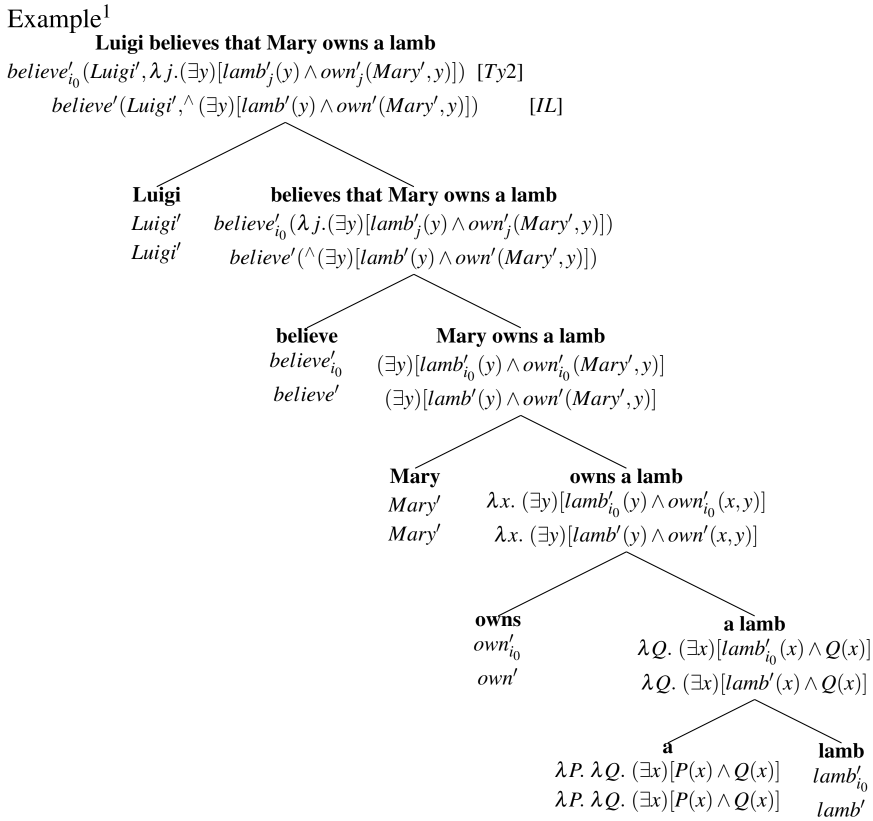
1 Two ways of representing intensionality

(1) Example[1]



(2) $Ty2$[2]

a. The set of *two-sorted types* is the smallest set $2T$ that contains (the primitive symbols) t, e, and s and is closed under pair formation; pairs (a, b) will sometimes be written without outermost parentheses and/or commas.

[1] collated from Zimmermann (2021)

[2] Cf. (Gallin, 1975: 58ff.), where the language is called Ty_2.

b. The language of *two-sorted type theory* – $Ty2$, for short – is the smallest family $(Ty2_a)_{a\in 2T}$ such that for any types $a, b \in 2T$:
$Con_a \subseteq Ty2_a$;
$Var_a \subseteq Ty2_a$;
whenever $\alpha \in Ty2_{ab}$ and $\beta \in Ty2_a$, $\ulcorner \alpha(\beta) \urcorner \in Ty2_b$;
whenever $x \in Var_a$ and $\alpha \in Ty2_b$, $\ulcorner (\lambda x.\alpha) \urcorner \in Ty2_{ab}$;
whenever α, $\beta \in Ty2_a$, $\ulcorner (\alpha = \beta) \urcorner \in Ty2_{\mathrm{t}}$.
The members of the sets $Ty2_a$ (where $a \in 2T$) will be referred to as the $Ty2$-*terms* of type a; the $Ty2$-terms of type t are also called $Ty2$-*formulae*.

c. A *two-sorted ontology* is a family $(D_a)_{a\in 2T}$ of non-empty sets such that for any a, $b \in 2T$:
$D_a = \{0,1\}$ if $a = \mathrm{t}$;
$D_{ab} = {D_b}^{D_a}$ [= the set of all total functions from D_a to D_b].

d. A $Ty2$-*model* is pair $(\mathscr{D}, F)$ where $\mathscr{D} = (D_a)_{a\in 2T}$ is a two-sorted ontology and $F : \bigcup Con \longrightarrow \bigcup D$ respects types, i.e.: $F(c) \in D_a$ whenever $c \in Con_a$; an assignment for a $Ty2$-model $(\mathscr{D}, F)$ is a function $g : \bigcup Var \longrightarrow \bigcup D$ that respects types.

e. If $(\mathscr{D}, F)$ is a $Ty2$-model, where $\mathscr{D} = (D_a)_{a\in 2T}$, and g is an assignment for M, then the *denotation* $[\![\alpha]\!]^{M,g} \in D_a$ of any $Ty2$-term α of any type $a \in 2T$ (*relative to M and g*) is defined by the following recursion:
$[\![\alpha]\!]^{M,g} = F(\alpha)$ if $\alpha \in \bigcup Con$;
$[\![\alpha]\!]^{M,g} = g(\alpha)$ if $\alpha \in \bigcup Var$;
$[\![\alpha]\!]^{M,g} = [\![\beta]\!]^{M,g}([\![\gamma]\!]^{M,g})$ if $\alpha = \ulcorner \beta(\gamma) \urcorner$, for some β, $\gamma \in \bigcup Ty2$;
$[\![\alpha]\!]^{M,g} = \{(u, [\![\beta]\!]^{M,g[^x/_u]}) \mid u \in D_b\}$ if for some $b, c \in 2T$, $\alpha = \ulcorner (\lambda x.\ \beta) \urcorner$,
where $x \in Var_b$, $\beta \in Ty2_c$, and $g[^x/_u] = [g \setminus \{(x, g(x))\}] \cup \{(x,u)\}$;
$[\![\alpha]\!]^{M,g} = \{v \in D_{\mathrm{t}} \mid v = 1 \text{ iff } [\![\beta]\!]^{M,g} = [\![\gamma]\!]^{M,g}\}$, if $\alpha = \ulcorner (\beta = \gamma) \urcorner$, for some β, $\gamma \in \bigcup Ty2$.

f. Logical symbols are defined as abbreviations; e.g. by:
$(\forall x)\varphi := ((\lambda x.\varphi) = (\lambda x.(x = x)))$, where $x \in \bigcup Var$ and $\varphi \in Ty2_{\mathrm{t}}$;
$\bot := (\forall x)x$, where $x \in Var_{\mathrm{t}}$
$= \quad ((\lambda x.\ x) = (\lambda x.\ (x = x)))$ ‘$\{1\} = D_{\mathrm{t}}$’
$\equiv \quad ((\lambda x.\ (\lambda y.\ (x = x))) = (\lambda x.\ (\lambda y.\ (x = y))))$ ‘$\bar{\bar{D}}_{\mathrm{t}} = 1$’ [also works for other types: see Sec. 2]
$[\neg\varphi] := (\varphi = \bot)$;
$[\varphi \wedge \psi] = (\lambda p.\ (\lambda q.\ (\lambda R.\ R(p)(q)) = (\lambda R.\ R([\neg\bot])([\neg\bot]))))(\varphi)(\psi)$,
where $p, q \in Var_{\mathrm{t}}$ and $R \in Var_{\mathrm{t(tt)}}$;
etc.

g. General logical notions like:
Freedom and bondage $[Fr(\alpha)]$;
substitution $[\alpha[^x/_\beta]]$];
closed terms (and formulae) $[Fr(\alpha) = \emptyset]$;
entailment $[\Sigma \vDash \varphi]$;
logical equivalence $[\equiv]$
etc.
are all defined in analogy to predicate logic.

h. *Alphabetic Variation* [aka α-*Conversion*]
$(\lambda x.\ \alpha) \equiv (\lambda y.\ \alpha[^x/y])$,
where y is a variable (of the same type as x) that does not occur in α.

i. η-*Conversion*
$(\lambda x.\ \alpha(x)) \equiv \alpha$ if $x \notin Fr(\alpha)$.

j. λ-*Conversion* [aka β-*Conversion*]
$(\lambda x.\ \alpha)(\beta) \equiv \alpha[^x/_\beta]$,
provided that no variable that is free in β gets bound in $\alpha[^x/_\beta]$. In particular:
$(\lambda x.\ \alpha)(x) \equiv \alpha$. ‘*eigen*-conversion’

(3) *IL*[3]

a. The set of *Fregean types* is the smallest set FT that contains (the primitive symbols) t and e, is closed under pair formation, and contains all pairs (s,a), where $a \in FT$.

b. The language of *intensional type logic* – *IL*, for short – is the smallest family $(IL_a)_{a\in FT}$ such that for any types $a, b \in FT$:

$Con_{\mathrm{s}a} \subseteq IL_a$;
$Var_a \subseteq IL_a$;
whenever $\alpha \in IL_{ab}$ and $\beta \in IL_a$, $\ulcorner\alpha(\beta)\urcorner \in IL_b$;
whenever $x \in Var_a$ and $\alpha \in IL_b$, $\ulcorner(\lambda x.\alpha)\urcorner \in IL_{ab}$;
whenever $\alpha, \beta \in IL_a$, $\ulcorner(\alpha = \beta)\urcorner \in IL_{\mathrm{t}}$;
whenever $\alpha \in IL_{\mathrm{s}b}$, $\ulcorner({}^{\vee}\alpha)\urcorner \in IL_b$;
whenever $\alpha \in IL_b$, $\ulcorner({}^{\wedge}\alpha)\urcorner \in IL_{\mathrm{s}b}$.

c. An *Fregean ontology* is a (unique) sub-family $(D_a)_{a\in FT\cup\{\mathrm{s}\}}$ of a two-sorted ontology $(D_a)_{a\in 2T}$.

d. An *IL-model* is pair $(\mathscr{D}, F)$ where $\mathscr{D} = (D_a)_{a\in FT\cup\{\mathrm{s}\}}$ is a Fregean ontology and $F : \bigcup_{a\in FT} Con_{\mathrm{s}a} \longrightarrow \bigcup D$ respects types; an assignment for an *IL*-model $(\mathscr{D}, F)$ is a function $g : \bigcup_{a\in FT} Var_a \longrightarrow \bigcup D$ that respects types.

e. If $(\mathscr{D}, F)$ is an *IL*-model, where $\mathscr{D} = (D_a)_{a\in FT\cup\{\mathrm{s}\}}$, g is an assignment for M, and $w \in D_{\mathrm{s}}$, then the *denotation* (or *extension*) $[\![\alpha]\!]^{M,g,w} \in D_a$ of any *IL*-term α of any type $a \in FT$ (*relative to* M, g, *and* w) is defined by the following recursion:

$[\![\alpha]\!]^{M,g,w} = F(\alpha)(w)$ if $\alpha \in \bigcup Con$;
$[\![\alpha]\!]^{M,g,w} = g(\alpha)$ if $\alpha \in \bigcup Var$;
$[\![\alpha]\!]^{M,g,w} = [\![\beta]\!]^{M,g,w}([\![\gamma]\!]^{M,g,w})$ if $\alpha = \ulcorner\beta(\gamma)\urcorner$, for some β, $\gamma \in \bigcup IL$;
$[\![\alpha]\!]^{M,g,w} = \{(u, [\![\beta]\!]^{M,g[^x/_u],w}) \mid u \in D_b\}$ if for some $b, c \in FT$, $\alpha = \ulcorner(\lambda x.\, \beta)\urcorner$,
where $x \in Var_b$, $\beta \in IL_c$, and $g[^x/_u] = [g \setminus \{(x, g(x))\}] \cup \{(x, u)\}$;
$[\![\alpha]\!]^{M,g,w} = \{v \in D_{\mathrm{t}} \mid v = 1 \text{ iff } [\![\beta]\!]^{M,g,w} = [\![\gamma]\!]^{M,g,w}\}$, if $\alpha = \ulcorner(\beta = \gamma)\urcorner$, for some β, $\gamma \in \bigcup IL$;
$[\![\alpha]\!]^{M,g,w} = [\![\beta]\!]^{M,g,w}(w)$ if $\alpha = \ulcorner({}^{\vee}\beta)\urcorner$, for some $\beta \in \bigcup IL$;
$[\![\alpha]\!]^{M,g,w} = \{w', [\![\beta]\!]^{M,g,w'}) \mid w' \in D_{\mathrm{s}}\}$ if for some $b \in FT$, $\alpha = \ulcorner({}^{\wedge}\beta)\urcorner$, where $\beta \in IL_b$.

The members of the sets IL_a (where $a \in FT$) will be referred to as the *IL-terms* of type a; the *IL*-terms of type t are also called *IL-formulae*.

f. Logical symbols are defined as abbreviations; e.g. by:

$[\Box\varphi] := (({}^{\wedge}\varphi) = ({}^{\wedge}[\neg\bot])$, where $\varphi \in IL_{\mathrm{t}}$;
$[\Diamond\varphi] := [\neg[\Box[\neg\varphi]]]$, where $\varphi \in IL_{\mathrm{t}}$;
etc.

g. General logical notions are as in (2g); in addition, an *IL*-term $\alpha \in \bigcup IL$ is *modally closed* if:[4]

$\alpha \in \bigcup Var$; or
$\alpha = \ulcorner({}^{\wedge}\beta)\urcorner$, for some $\beta \in \bigcup IL$; or
$\alpha \in \{\ulcorner\beta(\gamma)\urcorner, \ulcorner(\lambda x.\beta)\urcorner, \ulcorner(\beta = \gamma)\urcorner\}$, for some $x \in \bigcup Var$ and modally closed $\beta, \gamma \in \bigcup IL$.

(4) Gallin's * (aka *standard translation*)[5]

a. Any *IL*-term $\alpha \in IL_a$ (where $a \in FT$) translates into the $Ty2$-term $\alpha^* \in Ty2_a$ defined by the following recursion, where i_0 is a fixed variable of type s:

$\alpha^* = \ulcorner\alpha(i_0)\urcorner$, if $\alpha \in \bigcup Con$;
$\alpha^* = \alpha$, if $\alpha \in \bigcup Var$;
$\alpha^* = \ulcorner\beta^*(\gamma^*)\urcorner$, if $\alpha = \ulcorner\beta(\gamma)\urcorner$;
$\alpha^* = \ulcorner(\lambda x.\, \beta^*)\urcorner$, if $\alpha = \ulcorner(\lambda x.\, \beta)\urcorner$;
$\alpha^* = \ulcorner(\beta^* = \gamma^*)\urcorner$, if $\alpha = \ulcorner(\beta = \gamma)\urcorner$;
$\alpha^* = \ulcorner\beta^*(i_0)\urcorner$, if $\alpha = \ulcorner({}^{\vee}\beta)\urcorner$;
$\alpha^* = \ulcorner(\lambda i_0.\, \beta^*)\urcorner$, if $\alpha = \ulcorner({}^{\wedge}\beta)\urcorner$.

[3] Cf. (Montague, 1970: 384ff.), where the language is called L_0.
[4] (Gallin, 1975: 14)
[5] (Gallin, 1975: 61)

b. *Lemma* (Gallin, 1975: 14)
Any $\alpha \in IL_a$ (where $a \in FT$) is modally closed iff i_0 is not free in α^*.

c. *Lemma* (Gallin, 1975: 62)
For every *IL*-model $M = (\mathscr{D}, F)$ where $\mathscr{D} = (D_a)_{a \in FT \cup \{s\}}$, every *IL*- assignment g for M, and every $w \in D_s$ there is a *Ty*2-model $M^* = (\mathscr{D}, F^*)$ and a *Ty*2-assignment g^* for M^* such that for any *IL*-term $\alpha \in IL_a$ (where a):
$[\![\alpha]\!]^{M,g,w} = [\![\alpha^*]\!]^{M^*,g^*}$.

d. Hintikka semantics[6]
(i) **John knows that it is raining**
(ii) $(\forall j)[K_{i_0}(j)(John') \rightarrow \textit{it-is-raining}'_j]$
$\equiv$ $[\lambda F.\ (\forall j)[F(j)(John') \rightarrow \textit{it-is-raining}'_j]](K_{i_0})$ by λ-conversion (2j)
$\equiv$ $[\lambda x.[\lambda F.\ (\forall j)[F(j)(x) \rightarrow \textit{it-is-raining}'_j]](K_{i_0})](John')$ again by (2j)
$\equiv$ $[\lambda x.[\lambda F.\ (\forall i_0)[F(i_0)(x) \rightarrow \textit{it-is-raining}'_{i_0}]](K_{i_0})](John')$ by α-conversion (2h)
(iii) $[\lambda x.[\lambda F.\ \Box\ [^\vee F(x)\ \rightarrow\ \textit{it-is-raining}']](K)](John')$

e. A *Ty*2-term $\alpha \in Ty2_a$ is *Fregean* if $a \in FT$ and all variables occurring freely in α as well as all constants occurring in α are of Fregean types.

f. *Theorem* (Zimmermann, 1989: 75)
Any Fregean *Ty*2-term is logically equivalent to the translation of some *IL*-term.

g. Alphabetic variation (2h) and η-conversion (2i) work like in *Ty*2.

h. λ-conversion needs to be restricted though, to avoid clashes with the invisible variable:[7]
$(\lambda x.\alpha)(\beta) \equiv \alpha[^x/_\beta]$, provided that:
- no variable that is free in β gets bound in $\alpha[^x/_\beta]$, and:
- *either*: no free x in α lies in the scope of an intensional operator,
or: β is modally closed.

WARNING
IL's restricted version of λ-reduction does not allow for unique normal forms and is thus not a guaranteed route to readability:[8]
$(\lambda x.\ (\lambda y.\ (^\wedge\ y) = u(x))(x))(c)$
$\equiv$ $(\lambda y.\ (^\wedge\ y) = u(c))(c)$
$\equiv$ $(\lambda x.\ (^\wedge\ x) = u(x))(c)$
*Ty*2 : $(\lambda x.\ (\lambda y.\ (\lambda i_0.\ y) = u(x))(x))(c_{i_0})$
$\equiv$ $(\lambda j.\ c_{i_0}) = u(c_{i_0})$

i. In addition to (4h), there is another instance of *eigen*-conversion:
$(^\vee(^\wedge\alpha)) \equiv \alpha$ Down-up cancellation[9]

j. as well as of η-conversion:
$(^\wedge(^\vee\alpha)) \equiv \alpha$, if α is modally closed.

2 Non-standard semantics

(5) *IL* & *Ty*2: Lack of compactness and Löwenheim-Skolem

a. $\{(\exists^n x^e)(x = x) | n \in \mathbb{N}\} \vDash \varphi_\infty$ [= $(\exists M^{et})(\exists x^e)[[\neg M(x)] \wedge (\exists f^{ee} : D_e \longrightarrow M)$ 'f is one-one and onto']]]

b. If $[\![\varphi_\infty]\!]^{M,g[,w]} = 1$, then M is uncountable.

(6) Generalized semantics for *Ty*2

a. A *two-sorted g[eneralized]-ontology* is a family $(D_a)_{a \in 2T}$ of non-empty sets such that for any a, $b \in 2T$:
$D_a = \{0, 1\}$ if $a = \mathrm{t}$;
$D_{ab} \subseteq {D_b}^{D_a}$.

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[6] Hintikka (1969)
[7] (Gallin, 1975: 19)
[8] Friedman & Warren (1980: 323).
[9] (Dowty et al., 1981: 154)

b. A *Ty*2-*g-model* is a pair $(\mathscr{D}, F)$, where $\mathscr{D} = (D_a)_{a \in 2T}$ is a two-sorted g-ontology, $F : \bigcup_{a \in 2T} Con_a \longrightarrow \bigcup \mathscr{D}$ respects types, and there is a function $\mathscr{F} : \bigcup Ty2 \longrightarrow \cup \mathscr{D}$ such that for any g-assignment g and any $\alpha \in \bigcup Ty2, \mathscr{F}(\alpha) = [\![\alpha]\!]^{M,g}$, where:

$[\![\alpha]\!]^{M,g} = F(\alpha)$ if $\alpha \in \bigcup Con$;
$[\![\alpha]\!]^{M,g} = g(\alpha)$ if $\alpha \in \bigcup Var$;
$[\![\alpha]\!]^{M,g} = [\![\beta]\!]^{M,g}([\![\gamma]\!]^{M,g})$ if $\alpha = \ulcorner \beta(\gamma) \urcorner$, for some β, $\gamma \in \bigcup Ty2$;
$[\![\alpha]\!]^{M,g} = \{(u, [\![\beta]\!]^{M,g[^x/_u]}) \mid u \in D_b\}$ if for some $b, c \in 2T$, $\alpha = \ulcorner (\lambda x.\ \beta) \urcorner$,
where $x \in Var_b$, $\beta \in Ty2_c$, and $g[^x/_u] = [g \setminus \{(x, g(x))\}] \cup \{(x, u)\}$;
$[\![\alpha]\!]^{M,g} = \{v \in D_t \mid v = 1 \text{ iff } [\![\beta]\!]^{M,g} = [\![\gamma]\!]^{M,g}\}$, if $\alpha = \ulcorner(\beta = \gamma)\urcorner$, for some β, $\gamma \in \bigcup Ty2$.

c. <u>Axioms</u> for $Ty2$[10]

A1 $([g^{tt}([\neg\bot]) \wedge g^{tt}(\bot)] = (\forall x^t) g(x))$
A2 $(x^a = y^a) \rightarrow (P^{at}(x) = P(y))$
A3 $((\forall x^a)[f^{ab}(x) = g^{ab}(x)] = (f = g)$
AS4 = (2j)

<u>Rule of inference</u>

From $(\alpha = \beta)$ and φ to the result of replacing one occurrence of α in φ by β.

d. *Theorem* Andrews (1963)
The system in (6c) derives all g-valid *Ty2*-formulae.

e. *Corollary*
Any g-satisfiable set of $Ty2$-formulae has a g-model in which the domains of individuals and worlds are both at most denumerable.

(7) Generalized semantics for *IL*

a. A *Fregean g-ontology* is a (unique) sub-family $(D_a)_{a \in FT \cup \{s\}}$ of a two-sorted ontology $(D_a)_{a \in 2T}$.

b. An *IL-g-model* is a pair $(\mathscr{D}, F)$, where $\mathscr{D} = (D_a)_{a \in FT \cup \{s\}}$ is a Fregean g-ontology, $F : \bigcup_{a \in FT} Con_{sa} \longrightarrow \bigcup \mathscr{D}$ respects types, and there is a function $\mathscr{F} : \bigcup IL \longrightarrow \cup \mathscr{D}$ such that for any g-assignment g and any $\alpha \in \bigcup Ty2, \mathscr{F}(\alpha) = [\![\alpha]\!]^{M,g}$, where:

$[\![\alpha]\!]^{M,g,w} = F(\alpha)(w)$ if $\alpha \in \bigcup Con$;
$[\![\alpha]\!]^{M,g,w} = g(\alpha)$ if $\alpha \in \bigcup Var$;
$[\![\alpha]\!]^{M,g,w} = [\![\beta]\!]^{M,g,w}([\![\gamma]\!]^{M,g,w})$ if $\alpha = \ulcorner \beta(\gamma) \urcorner$, for some β, $\gamma \in \bigcup IL$;
$[\![\alpha]\!]^{M,g,w} = \{(u, [\![\beta]\!]^{M,g[^x/_u],w}) \mid u \in D_b\}$ if for some $b, c \in FT$, $\alpha = \ulcorner (\lambda x.\ \beta) \urcorner$,
where $x \in Var_b$, $\beta \in IL_c$, and $g[^x/_u] = [g \setminus \{(x, g(x))\}] \cup \{(x, u)\}$;
$[\![\alpha]\!]^{M,g,w} = \{v \in D_t \mid v = 1 \text{ iff } [\![\beta]\!]^{M,g,w} = [\![\gamma]\!]^{M,g,w}\}$, if $\alpha = \ulcorner(\beta = \gamma)\urcorner$, for some β, $\gamma \in \bigcup IL$;
$[\![\alpha]\!]^{M,g,w} = [\![\beta]\!]^{M,g,w}(w)$ if $\alpha = \ulcorner({}^\vee\beta)\urcorner$, for some $\beta \in \bigcup IL$;
$[\![\alpha]\!]^{M,g,w} = \{w', [\![\beta]\!]^{M,g,w'}) \mid w' \in D_s\}$ if for some $b \in FT$, $\alpha = \ulcorner({}^\wedge\beta)\urcorner$, where $\beta \in IL_b$.

c. <u>Axioms</u> for *IL*[11]

A1 $([g^{tt}([\neg\bot]) \wedge g^{tt}(\bot)] = (\forall x^t) g(x))$
A2 $(x^a = y^a) \rightarrow (P^{at}(x) = P(y))$
A3 $((\forall x^a)[f^{ab}(x) = g^{ab}(x)] = (f = g)$
AS4 = (4h)
A5 $(\Box({}^\vee f^{sa} = g^{sa}) = (f = g))$
AS6 = (4i)

<u>Rule of inference</u>

From $(\alpha = \beta)$ and φ to the result of replacing one occurrence of α in φ by β.

The system in (7c) derives all g-valid *IL*-formulae.

d. *Corollary*
Any g-satisfiable set of *IL*-formulae (of type t) has a g-model in which the domains of individuals and worlds are, respectively, at most denumerable and denumerable.

[10] Essentially Andrews (1963)
[11] (Gallin, 1975: 19)

(8) Reversing Gallin's *?[12]

a. It is unclear how [(4f)] can be extended to the generalized semantics for these languages [= *Ty*2 & *IL*] that has been discussed in [Gallin (1975)]. (Zimmermann, 1989: 75)

b. $((\lambda i.\,(\lambda j.\,(i=i))) = (\lambda i.\,(\lambda j.\,(i=j))))$ '$\bar{\bar{D}}_s = 1$': cf. (2f)

c. If $M = ((D_a)_{a \in 2T}, F)$ is a Ty2-g-model with any assignment g and $[\![(8b)]\!]^{M,g} = 1$, then $\bar{\bar{D}}_s = 1$.
$\Rightarrow$ There is no *IL*-formula φ such that (8b) is g-equivalent to φ^*: otherwise, φ would have a model $M = ((D_a)_{a \in FT}, F)$ with an infinite D_s, which would carry over to the *Ty*2-g-model $M^* = ((D_a)_{a \in FT}, F^*)$.

d. (8b)
$\equiv$ $((\lambda i.\,(\lambda j.\,[\neg\bot])) = (\lambda i.\,(\lambda j.((\lambda p^{st}.\,p(i)) = (\lambda p.\,p(j))))))$ in standard semantics
$\equiv$ $((\lambda i.\,(\lambda j.\,[\neg\bot])) = (\lambda i.(\lambda P^{(st)t}.\,(\lambda j.\,(P = (\lambda p.\,p(j)))))(((\lambda p^{st}.\,p(i))))))$
$\equiv$ $((\lambda i.\,(\lambda i.\,[\neg\bot])) = (\lambda i.(\lambda P^{(st)t}.\,(\lambda i.\,(P = (\lambda p.\,p(i)))))(((\lambda p^{st}.\,p(i))))))$
$\equiv$ $((^\wedge\,(^\wedge\,[\neg\bot])) = (^\wedge(\lambda P^{(st)t}.\,(^\wedge\,(P = (\lambda p.\,(^\vee p)))))(((\lambda p^{st}.\,{}^\vee p))))))^*$

e. *Theorem* [Walsh (soon)][13]
(4f) carries over to non-standard semantics if t is excluded as a ground type.

f. *Suggestion* (Walsh, pc [November 2024])
(4f) carries over to non-standard semantics if all g-models collapse into *simple* ones by merging equivalent worlds.[14]

g. Currying identity
(4f) carries over to non-standard semantics if identities $\ulcorner(\alpha = \beta)\urcorner$ in (2b) are re-analyzed as $\ulcorner Id(\alpha)(\beta)\urcorner$, where Id is a (logical) constant of type s(st):
(8b)
$\equiv$ $((\lambda i.\,(\lambda j.\,Id(i)(i))) = (\lambda i.\,(\lambda j.\,Id(i)(j))))$
$\equiv$ $((\lambda i.\,(\lambda j.\,Id(i)(i))) = (\lambda i.(\lambda p^{st}.\,(\lambda j.\,p(j)))(Id(i))))$
$\equiv$ $((\lambda i.\,(\lambda i.\,Id(i)(i))) = (\lambda i.(\lambda p^{st}.\,(\lambda i.\,p(i)))(Id(i))))$
$\equiv$ $((\lambda i.\,(\lambda i.\,Id(i)(i))) = (\lambda i.(\lambda p^{st}.\,(\lambda i.\,p(i)))(Id(i))))$
$\equiv$ $((^\wedge\,(^\wedge\,(^\vee Id))) = (^\wedge(\lambda p^{st}.\,(^\wedge\,(^\vee p)))(Id)))^*$
$A2^+(\lambda v^t.\,(\lambda q^{st}.\,\Box[^\vee q \to\ Id(v)(^\vee p^{st})]))(^\vee p)(Id)$

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[12] Zimmermann (2025)
[13] as announced in Walsh (2024) – if I got it right.
[14] See (Gallin, 1975: 80f.) for this model-theoretic technique applied to modal predicate logic.