Intensional type logic: standard translation and non-standard interpretation

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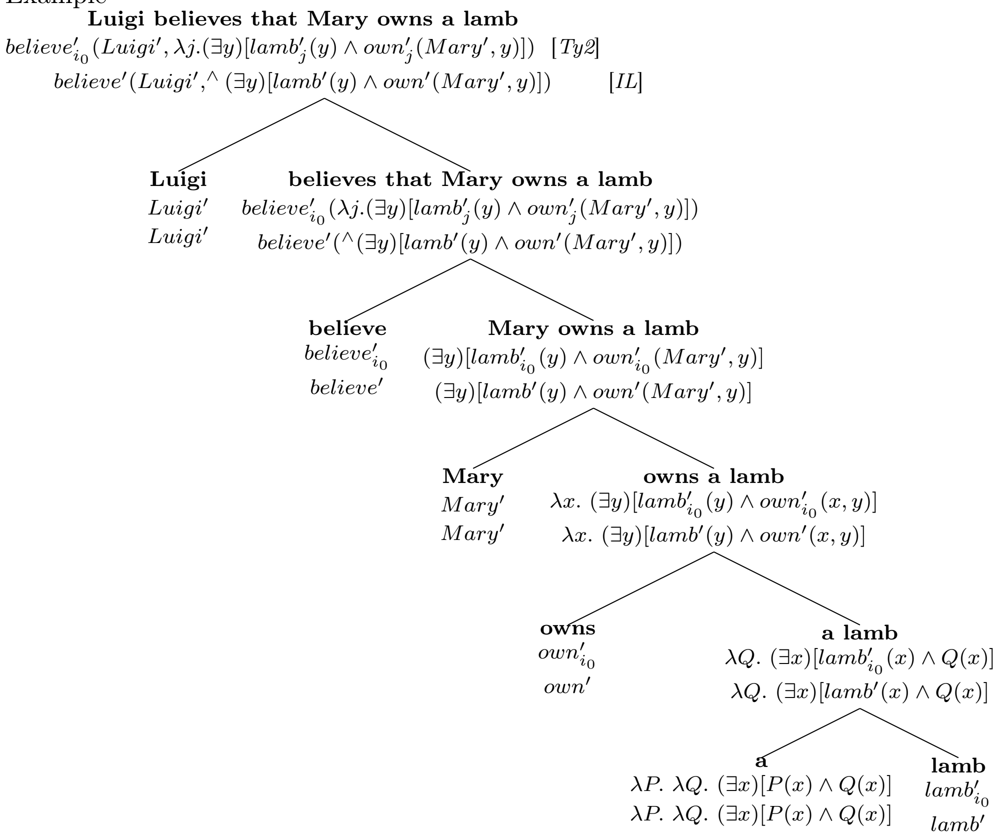
Abstract

Montague's (1970) intensional type logic (*IL*) has long been known to be largely expressively equivalent with its two-sorted substratum (*Ty2*). Zimmermann (1989) had shown that Gallin's (1975) standard translation * from *IL* to *Ty2* can be reversed so as to cover almost all of the latter, excepting only terms that are themselves of non-intensional types or contain such parameters. However the proof depended on the so-called *standard* interpretation of both languages, according to which abstraction and quantification range over full set-theoretic domains. The question of whether the result also holds for the restricted, completely axiomatisable version of Gallin (1975) had been open until recently. As it turns out, there may be more than one correct answer.

1 Two ways of representing intensionality

(1) Example[1]



[1] collated from Zimmermann (2021).

2 Two-sorted Type Theory[2]

(2) *Ty2*

a. The set of *two-sorted types* is the smallest set $2T$ that contains (the primitive symbols) t, e, and s and is closed under pair formation; pairs (a, b) will sometimes be written without outermost parentheses and/or commas.

b. The language of *two-sorted type theory* – *Ty2*, for short – is the smallest family $(Ty2_a)_{a \in 2T}$ such that for any types $a, b \in 2T$:

$Con_a \subseteq Ty2_a$;

$Var_a \subseteq Ty2_a$;

whenever $\alpha \in Ty2_{ab}$ and $\beta \in Ty2_a$, $\ulcorner \alpha(\beta) \urcorner \in Ty2_b$;

whenever $x \in Var_a$ and $\alpha \in Ty2_b$, $\ulcorner (\lambda x.\alpha) \urcorner \in Ty2_{ab}$;

whenever α, $\beta \in Ty2_a$, $\ulcorner (\alpha = \beta) \urcorner \in Ty2_{\mathrm{t}}$.

The members of the sets $Ty2_a$ (where $a \in 2T$) will be referred to as the *Ty2-terms* of type a; the *Ty2*-terms of type t are also called *Ty2-formulae.*

c. A *two-sorted ontology* is a family $(D_a)_{a \in 2T}$ of non-empty sets such that for any a, $b \in 2T$:

$D_a = \{0, 1\}$ if $a = \mathrm{t}$;

$D_{ab} = {D_b}^{D_a}$ [= the set of all total functions from D_a to D_b].

d. A *Ty2-model* is pair $(\mathcal{D}, \mathcal{F})$ where $\mathcal{D} = (\mathcal{D}_a)_{a \in 2T}$ is a two-sorted ontology and $\mathcal{F} : \bigcup Con \longrightarrow \bigcup \mathcal{D}$ respects types, i.e.: $\mathcal{F}(c) \in \mathcal{D}_a$ whenever $c \in Con_a$; an *assignment for* a *Ty2*-model $(\mathcal{D}, \mathcal{F})$ is a function $g : \bigcup Var \longrightarrow \bigcup \mathcal{D}$ that respects types.

e. If g is an assignment for a a $Ty2$-model $\mathcal{M} = (\mathcal{D}, \mathcal{F})$, then the *denotation* $[\![\alpha]\!]^{\mathcal{M},g} \in \mathcal{D}_a$ of any *Ty2*-term α of any type $a \in 2T$ (*relative to* $\mathcal{M}$ *and* g) is defined by the following recursion:

$[\![\alpha]\!]^{\mathcal{M},g} = \mathcal{F}(\alpha)$ if $\alpha \in \bigcup Con$;

$[\![\alpha]\!]^{\mathcal{M},g} = g(\alpha)$ if $\alpha \in \bigcup Var$;

$[\![\alpha]\!]^{\mathcal{M},g} = [\![\beta]\!]^{\mathcal{M},g}([\![\gamma]\!]^{\mathcal{M},g})$ if $\alpha = \ulcorner \beta(\gamma) \urcorner$, for some β, $\gamma \in \bigcup Ty2$;

$[\![\alpha]\!]^{\mathcal{M},g} = \{(\boldsymbol{u}, [\![\beta]\!]^{\mathcal{M},g[^x/_{\boldsymbol{u}}]}) \mid \boldsymbol{u} \in D_b\}$ if for some $b, c \in 2T$, $\alpha = \ulcorner (\lambda x.\ \beta) \urcorner$, where $x \in Var_b$, $\beta \in Ty2_c$, and $g[^x/_{\boldsymbol{u}}] = [g \setminus \{(x, g(x))\}] \cup \{(x, \boldsymbol{u})\}$;

$[\![\alpha]\!]^{\mathcal{M},g} = \{\boldsymbol{v} \in \mathcal{D}_{\mathrm{t}} \mid \boldsymbol{v} = 1 \text{ iff } [\![\beta]\!]^{\mathcal{M},g} = [\![\gamma]\!]^{\mathcal{M},g}\}$, if $\alpha = \ulcorner (\beta = \gamma) \urcorner$, for some β, $\gamma \in \bigcup Ty2$.

(3) Logic

a. Logical symbols are defined as abbreviations; e.g. by:

$(\forall x)\ \varphi := ((\lambda x.\ \varphi) = (\lambda x.\ (x = x)))$, where $x \in \bigcup Var$ and $\varphi \in Ty2_{\mathrm{t}}$;

$\bot := (\forall v)\ v$, where $v \in Var_{\mathrm{t}}$

$= ((\lambda v.\ v) = (\lambda v.\ (v = v)))$ ‘$\{1\} = D_{\mathrm{t}}$’

$\equiv ((\lambda x.\ (\lambda y.\ (x = x))) = (\lambda x.\ (\lambda y.\ (x = y))))$ ‘$\bar{\bar{D}}_{\mathrm{t}} = 1$’

[also works for other types: see (14c)]

$[\neg\varphi] := (\varphi = \bot)$;

$[\varphi \wedge \psi] = (\lambda p.\ (\lambda q.\ (\lambda R.\ R(p)(q)) = (\lambda R.\ R([\neg\bot])([\neg\bot]))))(\varphi)(\psi)$, where $p, q \in Var_{\mathrm{t}}$ and $R \in Var_{\mathrm{t(tt)}}$;

etc.

[2] Cf. (Gallin, 1975: 58ff.), where the language is called Ty_2. It is a version of the *simple theory of types*, which was introduced in Church (1940) as a deductive system with no formal interpretation.

b. General logical notions like:
Freedom and bondage $[Fr(\alpha)]$;
substitution $[\alpha[^x/_\beta]]]$;
closed terms (and formulae) $[Fr(\alpha) = \emptyset]$;
entailment $[\Sigma \vDash \varphi]$;
logical equivalence $[\equiv]$
etc.
are all defined in analogy to predicate logic.

(4) Laws of λ-conversion[3]

a. *α-Conversion* [aka Alphabetic Variation]
$(\lambda x.\ \alpha) \equiv (\lambda y.\ \alpha[^x/y])$,
where y is a variable (of the same type as x) that does not occur in α.

b. *η-Conversion*
$(\lambda x.\ \alpha(x)) \equiv \alpha$ if $x \notin Fr(\alpha)$.

c. *β-Conversion* [aka *λ-Conversion*]
$(\lambda x.\ \alpha)(\beta) \equiv \alpha[^x/_\beta]$,
provided that no variable that is free in β gets bound in $\alpha[^x/_\beta]$. In particular:
$(\lambda x.\ \alpha)(x) \equiv \alpha$. '*eigen*-conversion'

(5) Strong normalization[4]
Reduction [from left to right] by [successive] (4a), (4b), and (4c) always leads to irreducible terms that are unique up to renaming bound variables.

3 Intensional Type Logic

(6) *IL*[5]

a. The set of *Fregean types* is the smallest set FT that contains (the primitive symbols) t and e, is closed under pair formation, and contains all pairs (s,a), where $a \in FT$.

b. The language of *intensional type logic* – *IL*, for short – is the smallest family $(IL_a)_{a \in FT}$ such that for any types $a, b \in FT$:
$Con_{\mathrm{s}a} \subseteq ILa$;
$Var_a \subseteq IL_a$;
whenever $\alpha \in ILab$ and $\beta \in ILa$, $\ulcorner\alpha(\beta)\urcorner \in ILb$;
whenever $x \in Var_a$ and $\alpha \in ILb$, $\ulcorner(\lambda x.\alpha)\urcorner \in ILab$;
whenever α, $\beta \in ILa$, $\ulcorner(\alpha = \beta)\urcorner \in IL\mathrm{t}$;
whenever $\alpha \in IL\mathrm{s}b$, $\ulcorner({}^\vee\alpha)\urcorner \in ILb$;
whenever $\alpha \in ILb$, $\ulcorner({}^\wedge\alpha)\urcorner \in IL\mathrm{s}b$.
The members of the sets *ILa* (where $a \in FT$) will be referred to as the *IL-terms* of type a; the *IL*-terms of type t are also called *IL-formulae.*

c. A *Fregean ontology* is a sub-family $(\mathcal{D}_a)_{a \in FT \cup \{\mathrm{s}\}}[= \mathcal{D} \restriction FT \cup \{\mathrm{s}\}]$ of a two-sorted ontology $(\mathcal{D}_a)_{a \in 2T}$.

d. An *IL-model* is pair $(\mathcal{D}, \mathcal{F})$ where $\mathcal{D} = (\mathcal{D}_a)_{a \in FT \cup \{\mathrm{s}\}}$ is a Fregean ontology and $\mathcal{F} : \bigcup_{a \in FT} Con_{\mathrm{s}a} \longrightarrow \bigcup \mathcal{D}$ respects types; an assignment for an *IL*-model $(\mathcal{D}, \mathcal{F})$ is a function $g : \bigcup_{a \in FT} Var_a \longrightarrow \bigcup \mathcal{D}$ that respects types.

[3] Cf. (Hindley, 1997: ch. 1).
[4] Cf. (Hindley and Seldin, 2008: 294ff.).
[5] Cf. (Montague, 1970: 384ff.), where the language is called L_0.

e. If $\mathcal{M} = (\mathcal{D}, \mathcal{F})$ is an *IL*-model, where $\mathcal{D} = (\mathcal{D}_a)_{a \in FT \cup \{s\}}$, g is an assignment for $\mathcal{M}$, and $\boldsymbol{w} \in \mathcal{D}_s$, then the *denotation* (or *extension*) $[\![\alpha]\!]^{\mathcal{M},g,\boldsymbol{w}} \in \mathcal{D}_a$ of any *IL*-term α of any type $a \in FT$ (*relative to* $\mathcal{M}, g$, and $\boldsymbol{w}$) is defined by the following recursion:

$[\![\alpha]\!]^{\mathcal{M},g,\boldsymbol{w}} = \mathcal{F}(\alpha)(\boldsymbol{w})$ if $\alpha \in \bigcup Con$;

$[\![\alpha]\!]^{\mathcal{M},g,\boldsymbol{w}} = g(\alpha)$ if $\alpha \in \bigcup Var$;

$[\![\alpha]\!]^{\mathcal{M},g,\boldsymbol{w}} = [\![\beta]\!]^{\mathcal{M},g,\boldsymbol{w}}([\![\gamma]\!]^{\mathcal{M},g,\boldsymbol{w}})$ if $\alpha = \ulcorner\beta(\gamma)\urcorner$, for some β, $\gamma \in \bigcup IL$;

$[\![\alpha]\!]^{\mathcal{M},g,\boldsymbol{w}} = \{(\boldsymbol{u}, [\![\beta]\!]^{\mathcal{M},g[^x/_{\boldsymbol{u}}],\boldsymbol{w}}) \mid \boldsymbol{u} \in \mathcal{D}_b\}$ if for some $b, c \in FT$, $\alpha = \ulcorner(\lambda x.\ \beta)\urcorner$,
where $x \in Var_b$, $\beta \in ILc$, and $g[^x/_{\boldsymbol{u}}] = [g \setminus \{(x, g(x))\}] \cup \{(x, \boldsymbol{u})\}$;

$[\![\alpha]\!]^{\mathcal{M},g,\boldsymbol{w}} = \{\boldsymbol{v} \in \mathcal{D}_t \mid \boldsymbol{v} = 1 \text{ iff } [\![\beta]\!]^{\mathcal{M},g,\boldsymbol{w}} = [\![\gamma]\!]^{\mathcal{M},g,\boldsymbol{w}}\}$, if $\alpha = \ulcorner(\beta = \gamma)\urcorner$,
for some β, $\gamma \in \bigcup IL$;

$[\![\alpha]\!]^{\mathcal{M},g,\boldsymbol{w}} = [\![\beta]\!]^{\mathcal{M},g,\boldsymbol{w}}(\boldsymbol{w})$ if $\alpha = \ulcorner({}^\vee\beta)\urcorner$, for some $\beta \in \bigcup IL$;

$[\![\alpha]\!]^{\mathcal{M},g,\boldsymbol{w}} = \{(\boldsymbol{w'}, [\![\beta]\!]^{\mathcal{M},g,\boldsymbol{w'}}) \mid \boldsymbol{w'} \in D_s\}$ if for some $b \in FT$, $\alpha = \ulcorner({}^\wedge\beta)\urcorner$,
where $\beta \in ILb$.

f. Logical symbols are defined as in (3a); further symbols from modal logic may be introduced, e. g.:

$[\Box\varphi] := (({}^\wedge\varphi) = ({}^\wedge[\neg\bot]))$, where $\varphi \in IL$t;

$[\Diamond\varphi] := [\neg[\Box[\neg\varphi]]]$, where $\varphi \in IL$t; etc.

g. General logical notions are as in (3b); in addition, an *IL*-term $\alpha \in \bigcup IL$ is *modally closed* if:[6]

$\alpha \in \bigcup Var$; or

$\alpha = \ulcorner({}^\wedge\beta)\urcorner$, for some $\beta \in \bigcup IL$; or

$\alpha \in \{\ulcorner\beta(\gamma)\urcorner, \ulcorner(\lambda x.\beta)\urcorner, \ulcorner(\beta = \gamma)\urcorner\}$,
for some $x \in \bigcup Var$ and modally closed $\beta, \gamma \in \bigcup IL$.

h. In *IL*, alphabetic variation (4a) and η-conversion (4b) are just as valid as in *Ty2*, but:
β-conversion needs to be restricted, to avoid clashes with the invisible i_0:[7]

$(\lambda x.\alpha)(\beta) \equiv \alpha[^x/_\beta]$, provided that:

- no variable that is free in β gets bound in $\alpha[^x/_\beta]$, and:
- *either*: no free x in α lies in the scope of an intensional operator,
or: β is modally closed.

WARNING
IL's restricted version of λ-reduction does not allow for unique normal forms and is thus not a guaranteed route to readability:[8]

$(\lambda x.\ (\lambda y.\ ({}^\wedge\ y) = u(x))(x))(c)$
$\equiv(\lambda y.\ ({}^\wedge\ y) = u(c))(c)$
$\equiv(\lambda x.\ ({}^\wedge\ x) = u(x))(c)$
Ty2 :$(\lambda x.\ (\lambda y.\ (\lambda i_0.\ y) = u(x))(x))(c_{i_0})$
$\equiv(\lambda j.\ c_{i_0}) = u(c_{i_0})$

i. In addition to (6h), there is another instance of *eigen*-conversion:

$({}^\vee({}^\wedge\alpha)) \equiv \alpha$ Down-up cancellation[9]

j. as well as of η-conversion:

$({}^\wedge({}^\vee\alpha)) \equiv \alpha$, if α is modally closed.

[6] (Gallin, 1975: 14).
[7] (Gallin, 1975: 19).
[8] Friedman and Warren (1980: 323).
[9] (Dowty et al., 1981: 154).

4 Standard Translation

(7) Gallin's $*$[10]

a. Any *IL*-term $\alpha \in ILa$ (where $a \in FT$) translates into the *Ty2*-term $\alpha^* \in Ty2_a$ defined by the following recursion, where i_0 is a fixed variable of type s:

$\alpha^* = \ulcorner\alpha(i_0)\urcorner$, if $\alpha \in \bigcup Con$;
$\alpha^* = \alpha$, if $\alpha \in \bigcup Var$;
$\alpha^* = \ulcorner\beta^*(\gamma^*)\urcorner$, if $\alpha = \ulcorner\beta(\gamma)\urcorner$;
$\alpha^* = \ulcorner(\lambda x.\ \beta^*)\urcorner$, if $\alpha = \ulcorner(\lambda x.\ \beta)\urcorner$;
$\alpha^* = \ulcorner(\beta^* = \gamma^*)\urcorner$, if $\alpha = \ulcorner(\beta = \gamma)\urcorner$;
$\alpha^* = \ulcorner\beta^*(i_0)\urcorner$, if $\alpha = \ulcorner({}^\vee\beta)\urcorner$;
$\alpha^* = \ulcorner(\lambda i_0.\ \beta^*)\urcorner$, if $\alpha = \ulcorner({}^\wedge\beta)\urcorner$.

b. *Lemma* (Gallin, 1975: 14)
Any $\alpha \in ILa$ (where $a \in FT$) is modally closed iff i_0 is not free in α^*.

c. *-*Lemma* (Gallin, 1975: 62)
For every *IL*-model $\mathcal{M} = (\mathcal{D}, \mathcal{F})$ where $\mathcal{D} = (\mathcal{D}_a)_{a \in FT \cup \{s\}}$, every *IL*- assignment g for $\mathcal{M}$, and every $w \in D_s$ there is a *Ty2*-model $\mathcal{M}^* = (\mathcal{D}, \mathcal{F}^*)$ and a *Ty2*-assignment g^* for $\mathcal{M}^*$ such that for any *IL*-term $\alpha \in ILa$ (where $a \in FT$):

$[\![\alpha]\!]^{\mathcal{M},g,w} = [\![\alpha^*]\!]^{\mathcal{M}^*,g^*}$.

(8) Reversing Gallin's $*$[11]

a. Hintikka semantics[12]

(i) **John knows that it is raining**

(ii) $(\forall j)[K_{i_0}(j)(John') \rightarrow \textit{it-is-raining}'_j]$

$\equiv[\lambda F.\ (\forall j)[F(j)(John') \rightarrow \textit{it-is-raining}'_j]](K_{i_0})$ by β-conversion (4c)

$\equiv[\lambda x.[\lambda F.\ (\forall j)[F(j)(x) \rightarrow \textit{it-is-raining}'_j]](K_{i_0})](John')$ again by (4c)

$\equiv[\lambda x.[\lambda F.\ (\forall i_0)[F(i_0)(x) \rightarrow \textit{it-is-raining}'_{i_0}]](K_{i_0})](John')$ by α-conversion (4a)

(iii) $[\lambda x.[\lambda F.\ \Box\ [{}^\vee F(x)\ \rightarrow\ \textit{it-is-raining}']](K)](John')$

b. $\alpha \in Ty2_a$ is *Fregean* if $a \in FT$, $Fr(\alpha) \subseteq \{i_0\} \cup \bigcup_{b \in FT} Var_b$, and all constants occurring in α are of Fregean types of the form (sb).

c. *Theorem* (Zimmermann, 1989: 75)
Any Fregean *Ty2*-term α is logically equivalent to the translation of an *IL*-term α_*:

${\alpha_*}^* \equiv \alpha$.

5 Non-standard interpretation

(9) *IL* & *Ty2*: Lack of compactness and Löwenheim-Skolem

a. $\{(\exists^n x^e)(x = x) | n \in \mathbb{N}\} \models \varphi_\infty$

$[= (\exists M^{et})(\exists x^e)[[\neg M(x)] \wedge (\exists f^{ee} : D_e \longrightarrow M)$ 'f is one-one and onto'$]]$

b. If $[\![\varphi_\infty]\!]^{M,g[,w]} = 1$, then M is uncountable.

(10) Generalized interpretation of *Ty2*

a. A *two-sorted g[eneralized]-ontology* is a family $(D_a)_{a \in 2T}$ of non-empty sets such that for any $a, b \in 2T$:

$D_a = \{0, 1\}$ if $a =$ t;
$D_{ab} \subseteq {D_b}^{D_a}$.

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[10] (Gallin, 1975: 61).
[11] Cf. Zimmermann (2021), plus Köpping and Zimmermann (2020) for more.
[12] Hintikka (1969).

b. A *Ty2-g-model* is a pair $(\mathcal{D}, F)$, where $\mathcal{D} = (D_a)_{a \in 2T}$ is a two-sorted g-ontology, $F : \bigcup_{a \in 2T} Con_a \longrightarrow \bigcup \mathcal{D}$ respects types, and there is a function $\mathcal{F} : \bigcup Ty2 \longrightarrow \cup \mathcal{D}$ such that for any g-assignment g and any $\alpha \in \bigcup Ty2, \mathcal{F}(\alpha) = [\![\alpha]\!]^{M,g}$, where:

$[\![\alpha]\!]^{M,g} = F(\alpha)$ if $\alpha \in \bigcup Con$;
$[\![\alpha]\!]^{M,g} = g(\alpha)$ if $\alpha \in \bigcup Var$;
$[\![\alpha]\!]^{M,g} = [\![\beta]\!]^{M,g}([\![\gamma]\!]^{M,g})$ if $\alpha = \ulcorner\beta(\gamma)\urcorner$, for some β, $\gamma \in \bigcup Ty2$;
$[\![\alpha]\!]^{M,g} = \{(u, [\![\beta]\!]^{M,g[^x/_u]}) \mid u \in D_b\}$ if for some $b, c \in 2T$, $\alpha = \ulcorner(\lambda x.\ \beta)\urcorner$, where $x \in Var_b$, $\beta \in Ty2_c$, and $g[^x/_u] = [g \setminus \{(x, g(x))\}] \cup \{(x, u)\}$;
$[\![\alpha]\!]^{M,g} = \{v \in D_{\mathrm{t}} \mid v = 1$ iff $[\![\beta]\!]^{M,g} = [\![\gamma]\!]^{M,g}\}$, if $\alpha = \ulcorner(\beta = \gamma)\urcorner$, for some β, $\gamma \in \bigcup Ty2$.

(11) Axiomatising *Ty2*

a. Axioms[13]

A1 $([g^{\mathrm{tt}}([\neg\bot]) \wedge g^{\mathrm{tt}}(\bot)] = (\forall x^{\mathrm{t}})g(x))$
A2 $(x^a = y^a) \to (P^{a\mathrm{t}}(x) = P(y))$
A3 $((\forall x^a)[f^{ab}(x) = g^{ab}(x)] = (f = g)$
AS4 = (4c)

Rule of inference

From $(\alpha = \beta)$ and φ to the result of replacing one occurrence of α in φ by β.

b. *Theorem* (Andrews (1963))
The system in (11a) derives all g-valid *Ty2*-formulae.

c. *Corollary*
Any g-satisfiable set of *Ty2*-formulae has a g-model in which the domains of individuals and worlds are both at most denumerable.

(12) Generalized interpretation of *IL*

a. A *Fregean g-ontology* is a (unique) sub-family $(D_a)_{a \in FT \cup \{\mathrm{s}\}}$ of a two-sorted ontology $(D_a)_{a \in 2T}$.

b. An *IL-g-model* is a pair $(\mathcal{D}, F)$, where $\mathcal{D} = (D_a)_{a \in FT \cup \{\mathrm{s}\}}$ is a Fregean g-ontology, $F : \bigcup_{a \in FT} Con_{\mathrm{s}a} \longrightarrow \bigcup \mathcal{D}$ respects types, and there is a function $\mathcal{F} : \bigcup IL \longrightarrow \cup \mathcal{D}$ such that for any g-assignment g and any $\alpha \in \bigcup Ty2, \mathcal{F}(\alpha) = [\![\alpha]\!]^{M,g}$, where:

$[\![\alpha]\!]^{M,g,w} = F(\alpha)(w)$ if $\alpha \in \bigcup Con$;
$[\![\alpha]\!]^{M,g,w} = g(\alpha)$ if $\alpha \in \bigcup Var$;
$[\![\alpha]\!]^{M,g,w} = [\![\beta]\!]^{M,g,w}([\![\gamma]\!]^{M,g,w})$ if $\alpha = \ulcorner\beta(\gamma)\urcorner$, for some β, $\gamma \in \bigcup IL$;
$[\![\alpha]\!]^{M,g,w} = \{(u, [\![\beta]\!]^{M,g[^x/_u],w}) \mid u \in D_b\}$ if for some $b, c \in FT$, $\alpha = \ulcorner(\lambda x.\ \beta)\urcorner$, where $x \in Var_b$, $\beta \in ILc$, and $g[^x/_u] = [g \setminus \{(x, g(x))\}] \cup \{(x, u)\}$;
$[\![\alpha]\!]^{M,g,w} = \{v \in D_{\mathrm{t}} \mid v = 1$ iff $[\![\beta]\!]^{M,g,w} = [\![\gamma]\!]^{M,g,w}\}$, if $\alpha = \ulcorner(\beta = \gamma)\urcorner$, for some β, $\gamma \in \bigcup IL$;
$[\![\alpha]\!]^{M,g,w} = [\![\beta]\!]^{M,g,w}(w)$ if $\alpha = \ulcorner({}^\vee\beta)\urcorner$, for some $\beta \in \bigcup IL$;
$[\![\alpha]\!]^{M,g,w} = \{w', [\![\beta]\!]^{M,g,w'}) \mid w' \in D_{\mathrm{s}}\}$ if for some $b \in FT$, $\alpha = \ulcorner({}^\wedge\beta)\urcorner$, where $\beta \in ILb$.

(13) Axiomatising *IL*

a. Axioms for *IL*[14]

A1–A3 as in (11a) PLUS:
AS4 = (6h)
A5 $(\Box(({}^\vee f^{\mathrm{s}a}) = ({}^\vee g^{\mathrm{s}a})) = (f = g))$ [≈ (A3)]
AS6 = (6i)

Rule of inference

From $(\alpha = \beta)$ and φ to the result of replacing one occurrence of α in φ by β.

[13] Essentially Andrews (1963).
[14] (Gallin, 1975: 19).

b. *Theorem* (Gallin, 1975: 30)
The system in (13a) derives all g-valid *IL*-formulae.

c. *Corollary*
Any g-satisfiable set of *IL*-formulae has a g-model in which the domains of individuals and worlds are, respectively, at most denumerable and denumerable.

(14) Reversing Gallin's *?[15]

a. Gallin's *-Lemma (7c) carries over to the non-standard interpretation of *IL* and *Ty2*. However:
QUOTE It is unclear how [(8c)] can be extended to the generalized interpretation for these languages [= *Ty2* & *IL*] that has been discussed in [Gallin (1975)]. UNQUOTE
(Zimmermann, 1989: 75)

b. Disambiguating (14a)
A If α is any Fregean *Ty2*-term, is ${\alpha_*}^* \equiv_g \alpha$, where α_* is as in (8c)?
B If α is any Fregean *Ty2*-term, is there an *IL*-term α_+ such that ${\alpha_+}^* \equiv_g \alpha$?
C For which Fregean *Ty2*-terms α is there an *IL*-term α_+ such that ${\alpha_+}^* \equiv_g \alpha$?

c. *ad* **A** & **B**:
σ
$:=$ $((\lambda i.\ (\lambda j.\ (i = i))) = (\lambda i.\ (\lambda j.\ (i = j))))$ '$\bar{\bar{D}}_s = 1$': cf. (3a)
$\equiv$ $((\lambda i.\ (\lambda j.\ [\neg\bot])) = (\lambda i.\ (\lambda j.((\lambda p^{st}.\ p(i)) = (\lambda p.\ p(j))))))$ in standard interpretation
$\equiv$ $((\lambda i.\ (\lambda j.\ [\neg\bot])) = (\lambda i.(\lambda P^{(st)t}.\ (\lambda j.\ (P = (\lambda p.\ p(j)))))(((\lambda p^{st}.\ p(i))))))$
$\equiv$ $((\lambda i.\ (\lambda i.\ [\neg\bot])) = (\lambda i.(\lambda P^{(st)t}.\ (\lambda i.\ (P = (\lambda p.\ p(i)))))(((\lambda p^{st}.\ p(i))))))$
$\equiv$ $((^\wedge\ (^\wedge\ [\neg\bot])) = (^\wedge(\lambda P^{(st)t}.\ (^\wedge\ (P = (\lambda p.\ (^\vee p)))))(((\lambda p^{st}.\ {}^\vee p))))))^*$
If $[\![\sigma]\!]^{\mathcal{M},g} = 1$ (for any *Ty2*-g-model $\mathcal{M} = ((\mathcal{D}_a)_{a\in 2T}, F)$ and assignment g), then $\bar{\bar{D}}_s = 1$.
$\Rightarrow$ There is no *IL*-formula σ_+ such that ${\sigma_+}^* \equiv_g \sigma$ ...
Otherwise, σ_+ would have a g-model $\mathcal{M} = ((D_a)_{a\in FT}, F)$ with an infinite $\mathcal{D}_s$ and so, by the *-lemma, $\mathcal{M}^*$ would be a g-model of ${\sigma_+}^*$; but $\mathcal{M}$ and $\mathcal{M}^*$ share the infinite domain $\mathcal{D}_s$, and so $[\![\sigma]\!]^{\mathcal{M}^*} = 0$.
... which means that the answer to **B**, and *a fortiori* to **A**, is **NO**.

d. *ad* **C**:
Theorem (Walsh (soon))[16]
(8c) carries over to non-standard interpretation if t is excluded as a ground type.
... thus providing a partial answer to **C**.

e. *Suggestion* (Sean Walsh, pc, November 2024)
(8c) carries over to non-standard interpretation if all g-models collapse into *simple* ones by merging equivalent worlds.[17]

f. *Conjecture*
(8c) carries over to non-standard interpretationcs if Id is a (logical) constant of type s(st) and the following clause is added to (8c):
$[\![Id]\!]^{M,g,\boldsymbol{w}}(\boldsymbol{w'}) = 1$ iff $\boldsymbol{w} = \boldsymbol{w'}$, for any $\boldsymbol{w'} \in D_s$.
σ
$\equiv$ $((\lambda i.\ (\lambda j.\ Id(i)(i))) = (\lambda i.\ (\lambda j.\ Id(i)(j))))$
$\equiv$ $((\lambda i.\ (\lambda j.\ Id(i)(i))) = (\lambda i.(\lambda p^{st}.\ (\lambda j.\ p(j)))(Id(i))))$
$\equiv$ $((\lambda i.\ (\lambda i.\ Id(i)(i))) = (\lambda i.(\lambda p^{st}.\ (\lambda i.\ p(i)))(Id(i))))$
$\equiv$ $((\lambda i.\ (\lambda i.\ Id(i)(i))) = (\lambda i.(\lambda p^{st}.\ (\lambda i.\ p(i)))(Id(i))))$
$=$ $((^\wedge\ (^\wedge\ (^\vee Id))) = (^\wedge(\lambda p^{st}.\ (^\wedge\ (^\vee p)))(Id)))^*$

g. Additional axiom
$A2^+(\lambda v^t.\ (\lambda q^{st}.\ \Box[^\vee q \rightarrow\ Id(v)(^\vee p^{st})]))(^\vee p)(Id)$

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[15] Zimmermann (2025).
[16] as announced in Walsh (2025).
[17] See (Gallin, 1975: 80f.) for this model-theoretic technique, where it is applied to modal predicate logic.

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