From Frege to Bäuerle

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0. *Frege's Doctrine and Bäuerle's Problem*

• *Frege's doctrine*

Frege believed that the referent of a sentence is its truth-value, the True or the False. The *Gedanke* or thought expressed is its *Sinn*. The referent of "that the planetary orbits are circles" is the *Gedanke*; that is what is believed. But since the *Gedanke* is the *Sinn* of the sentence, its components are themselves *Sinne*, and hence must be the references of the component parts of the sentence following the "that" clause. [footnote omitted]

So there is a reference shift in indirect contexts. Since the reference must be determined in some way, there has to be therefore not only an indirect reference but also apparently an indirect sense. As far as I know Frege does not ever explicitly consider iterations of "that" clauses, such as, for example, "We should remember that Copernicus believed that the planetary orbits are circles", and so on for arbitrary iterations. If we do consider such iterations, the familiar consequence is that Frege is committed to a hierarchy of doubly oblique indirect referents and senses, triply oblique, and so on. Kripke (2008: 183)

If words are used in the ordinary way, one intends to speak of their referents. It can also happen, however, that one wishes to talk about the words themselves or their sense. […] In reported speech one talks about the sense – e.g., of another person's remarks. It is quite clear that in this way of speaking words do not have their customary referents but designate what is usually their sense. In order to have a short expression, we will say: In reported speech, words are used indirectly or have their indirect referents. We distinguish accordingly the customary from the indirect referent of a word; and its customary sense from its indirect sense. The indirect referent of a word is accordingly its customary sense. Such exceptions must always be borne in mind if the mode of connection between sign, sense, and referent in particular cases is to be correctly understood. Frege (1948: 211f.) [1892: 28]

• *Bäuerle's problem* — Bäuerle (1983), Percus (2001), Keshet(2010*a*, *b*)

[a] All national team members are staying in a 5-star hotel.
ought to be ambiguous between:
[b] For all national team members x it holds that there is a 5-star hotel y such that x is staying in y.
[c] There is a 5-star hotel y such that for all national team members x it holds that x is staying in y.
Now, if [a] on the intended reading [c], is embedded in a belief context, as in:
[d] George believes that all national team members are staying in a 5-star hotel.
then it is not possible to have the universally quantified NP transparent and the other one opaque by scope, because such a reading would give the universal quantifier wide scope over the existential quantifier, contradicting [c]. But why should it not be part of George's belief to believe of certain gentlemen (who George himself does not as the national team members) that there is exactly one hotel that they are staying in?
[…] Moreover, the problem is not at all restricted to belief contexts but quite generally concerns quantifying into non-extensional contexts. Bäuerle (1983: 124) [translation by TEZ]

• *…and related phenomena*

<u>Singular *de re*</u>

Scotland Yard are carrying out an investigation into the murder of a music producer at a recording studio and have arrested a suspect, who they take to be a hit man though he is actually a hit composer. Around the time of murder Syd, the janitor, had seen a person leaving the studio in a rush, and so the police conduct an identity parade with the suspect and five innocent police officers – one of whom Syd picks out with confidence. Detective inspector Norman, who arrives late at the line-up, mistakes his colleague for the suspect, and thus:

(1) Norman thinks that Syd saw that the hit man had left the studio in a rush.

<u>Plural *de re[bus]*</u>

Scotland Yard are carrying out an investigation into the murder of a music producer at a recording studio and have arrested two suspects, who they take to be hit men though they are actually hit composers. Around the time of murder Syd, the janitor, had seen two persons leaving the studio in a rush, and so the police conduct an identity parade with the suspects and ten innocent police officers – two of whom Syd picks out with confidence. Detective inspector Norman, who arrives late at the line-up, mistakes his colleagues for the suspects, and thus:

(2) Norman thinks that Syd saw that the hit men had left the studio in a rush.

Intermediate scope indefinites — Abusch (1993)

Scotland Yard are carrying out an investigation into the murder of a music producer at a recording studio and have arrested two suspects, who they take to be hit men though they are actually hit composers. Around the time of murder Syd, the janitor, had seen a person leaving the studio in a rush, and so the police conduct an identity parade with the suspects and ten innocent police officers – two of whom Syd picks out with confidence. Detective inspector Norman, who arrives late at the line-up, mistakes his colleague for the suspect, and thus:

(3) Norman thinks that Syd saw that a hit man had left the studio in a rush.

Bäuerle reading

Scotland Yard are carrying out an investigation into the murder of a music producer at a recording studio and have arrested two suspects, who they take to be hit men though they are actually hit composers. Around the time of murder Syd, the janitor, had seen a person leaving the studio in a rush, and so the police conduct an identity parade with the suspects and ten innocent police officers – two of whom Syd picks out with confidence without being able to decide between them. Detective inspector Norman, who arrives late at the line-up, mistakes his colleagues for the suspects, and thus:

(4) Norman thinks that Syd saw that a hit man had left the studio in a rush.

(5) Syd sees that every band member is drinking.

(5*a*) $\mathbf{S}_i(\mathbf{s}, \lambda \boldsymbol{j}.\ \forall(\mathbf{B}_j)(\mathbf{D}_j))$ — *in situ*

(*b*) $\forall(\mathbf{B}_i)\ (\lambda \boldsymbol{x}.\mathbf{S}_i(\mathbf{s}, \lambda \boldsymbol{j}.\ \mathbf{D}_j(\boldsymbol{x})))$ — wide scope

(*c*) $\mathbf{S}_i(\mathbf{s}, \lambda \boldsymbol{j}.\ \forall(\mathbf{B}_i)(\mathbf{D}_j))$ — Bäuerle reading

Descriptive generalisation

- $\ldots [\lambda i_m. \ldots\ \lambda i_n. \ldots [\![X]\!]^{i_n}) \ldots]$ — *in situ*

  $\ldots [\lambda i_m. \ldots\ \lambda i_n. \ldots [\![X]\!]^{i_m}) \ldots]$ — Saarinen (1979), Cresswell (1990), Schlenker (2006)

Generalisation X — Percus (2001: 201)

(47*a*) Syd said that every neighbour of Emily's <u>is a band member</u>.

(*b*) $S_i(\mathrm{s}, \lambda j.\ \forall(N_j(\mathrm{e}), B_i))$ — unattested reading

Bäuerle effect

If

(β_1) $\ldots [\lambda i_m. \ldots\ \lambda i_n. \ldots [\![\delta]\!]([\![R]\!]^{i_n}, [\![S]\!]^{i_n}) \ldots]$ — *in situ*

is a reading, then so is:

(β_2) $\ldots [\lambda i_m. \ldots\ \lambda i_n. \ldots [\![\delta]\!]([\![R]\!]^{i_m}, [\![S]\!]^{i_n}) \ldots]$ — Bäuerle constellation

1. ***Semantic values: the theory of extension and intension*** Carnap (1947), Montague (1970)

Rough characterisation

- Every expression has (at least) two semantic values, its extension and its intension, which capture two communicative functions of linguistic meaning, reference and content.
- The intension of an expression comprises its possible extensions, its extension is relative to a point of reference in Logical Space.
- If compositionality demands it, the intension may replace the extension. Fregean compositionality

Two versions

- **Strong**: Extensions are (stand-ins for) referents; intensions are (stand-ins for) content.

(6*a*) $ext_w(Gottlob) = \text{Frege} = \rho_w(Gottlob)$ referent

(*b*) $ext_w(Bertie) = \{A \subseteq D \mid \text{Russell} \in A\}$; $\rho_w(Bertie) = \iota(\bigcap(ext_w(Bertie)))$ stand-in

- **Weak**: Extensions help determining the referents of (possibly other, larger) expressions; intensions help determining the information values of (possibly other, larger) expressions.

(7*a*) $int(earnest) = ext_w(the\ property\ of\ being\ earnest) = P$ internal value

(*b*) $\rho_w(the\ property\ of\ being\ earnest) = P$; $earnest \notin dom(\rho_w)$ external value

2. ***Indirect interpretation***

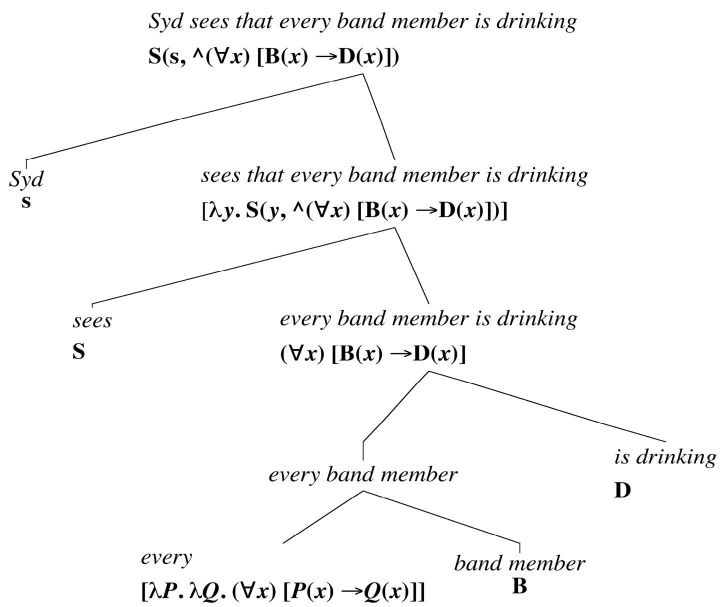
… with parameters (IL) Montague (1970)

$|\ earnest\ | = \mathbf{E} \in Con_{\boldsymbol{e}\,\boldsymbol{t}}$

Characteristics

- Translations of expressions denote extensions relative to points in Logical Space.
- Intensions can be derived by functional abstraction from these points of reference expressed by the cap operator '^':
- The domain of denotation includes extensions of all types: truth values (type ***t***), individuals (***e***), functional extensions (*ab*) assigning extensions (of types *a*) to extensions (of types *b*), as well as intensions (*sa*) – but not the points of Logical Space, or any functions into Logical Space.

(8)



… with variables (Ty2) Groenendijk & Stokhof (1982)

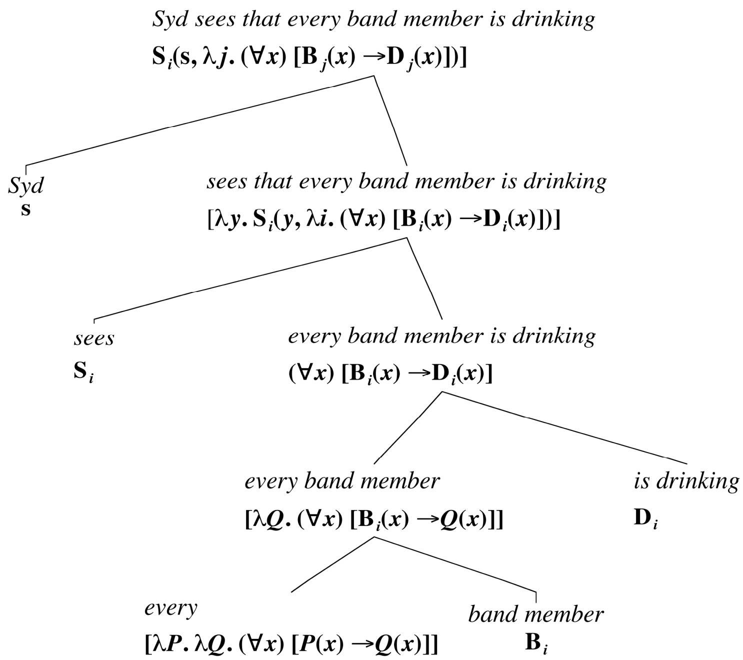
| *earnest* | = $\mathbf{E}(\boldsymbol{i})$, where $\mathbf{E} \in Con_{s(et)}$

$[\![\mathbf{E}(\boldsymbol{i})]\!]^{g,\dots} = P(\mathrm{g}(\boldsymbol{i})) \subseteq D$

Characteristics

- Translations of expressions are open formulae with a free variable $\boldsymbol{i} \in Var_s$ denoting the point of reference.
- Intensions can be derived by applying λ-abstraction to this variable.
- Apart from the extensions, the domain of denotation includes objects of *surplus* types: points of Logical Space (***s***), and functions from denotations to denotations (e.g., ***(se)s***).

(9)

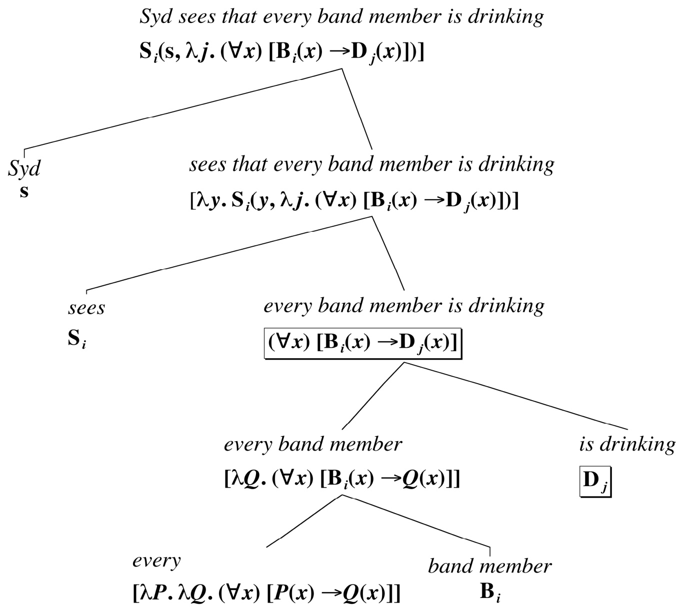


Comparison

- Any *IL*-formula α may be expressed by a *Ty2*-formula α*, in which the fixed *Ty2*-variable ***i*** represents the point of reference.
- Not every *Ty2*-formula β is (equivalent to) some α*. In particular, *(a)* β may contain free variables of type ***s*** other than ***i***; *(b)* β may contain free variables or constants of surplus types; *(c)* β may itself be of a surplus type.
- Formulae β that satisfy one of *(a)–(c)* do not denote extensions and are thus *irrelevant* to the theory of extension and intension.
- All relevant *Ty2*-formulae can be expressed in *IL*. Gallin (1975) [for type ***t***], Zimmermann (1989)
- A *Ty2*-translation conforms to the theory of extension and intension iff it is equivalent to an *IL*-formula.

3. *Bäuerle constellations in indirect interpretation*

(10)

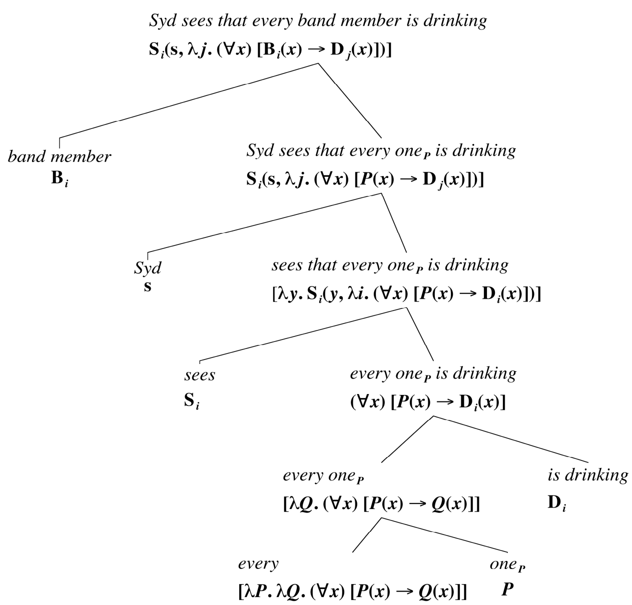


uninterpretable values?

Zimmermann (2012)

Restrictors as res — Groenendijk & Stokhof (1982)

(11)



4. *A new approach to Bäuerle's problem*

(12*a*) Norman hears that Syd sees that Emily is playing.

(*b*) $\mathfrak{P}(\mathit{Norman}, \mathfrak{A}(\mathit{hear}, \mathfrak{P}(\mathit{Syd}, \mathfrak{A}(\mathit{see}, \mathfrak{P}(\mathit{Emily}, \mathit{play})))))$

(*f*) $[\![\mathfrak{P}(\mathit{Norman}, \mathfrak{A}(\mathit{hear}, \mathfrak{P}(\mathit{Syd}, \mathfrak{A}(\mathit{see}, \mathfrak{P}(\mathit{Emily}, \mathit{play})))))]\!]^{i_0}$ Frege's doctrine

$= [\![\mathfrak{P}]\!]_{*}([\![\mathit{Norman}]\!]^{i_0}, [\![\mathfrak{A}]\!]_{*}([\![\mathit{hear}]\!]^{i_0}, [\![\mathfrak{P}]\!]_{**}([\![\mathit{Syd}]\!]^{\wedge}, [\![\mathfrak{A}]\!]_{**}([\![\mathit{see}]\!]^{\wedge}, [\![\mathfrak{P}]\!]_{***}([\![\mathit{Emily}]\!]^{\wedge\wedge}, [\![\mathit{play}]\!]^{\wedge\wedge})))))$

The hierarchy of rigid indirect intensions Parsons (1981)

0: $[\![\mathit{Emily}]\!]^{i_0} = \mathrm{e}$; $[\![\mathit{play}]\!]^{i_0} = P_{i_0}$; … extensions proper (at given i_0)

1: $[\![\mathit{Emily}]\!]^{\wedge} = \lambda i.\, \mathrm{e}$; $[\![\mathit{play}]\!]^{\wedge} = \lambda i.\, P_i$; … intensions

2: $[\![\mathit{Emily}]\!]^{\wedge\wedge} = \lambda i.\, \lambda j.\, \mathrm{e}$; $[\![\mathit{play}]\!]^{\wedge} = \lambda i.\, \lambda j. P_j$; … in-intensions

3: $[\![\mathit{Emily}]\!]^{\wedge\wedge\wedge} = \lambda i.\, \lambda j.\, \lambda k.\, \mathrm{e}$; $[\![\mathit{play}]\!]^{\wedge} = \lambda i.\, \lambda j.\, \lambda k.\, P_k$; … in-in-intensions

…

(13*a*) Norman hears that Syd sees that every band member is drinking.

(*b*) $\mathbf{H}^0(\mathbf{n}, {}^\wedge\, \mathbf{S}^1({}^\wedge\mathbf{s}, {}^{\wedge\wedge}\, \forall(\mathbf{B})(\mathbf{D})]))$ Zimmermann (2015): 'baroque compositionality'

$\equiv \mathbf{H}(\mathbf{n}, {}^\wedge\, \mathbf{S}(\mathbf{s}, {}^\wedge\, \forall(\mathbf{B})(\mathbf{D})]))$ where $\mathbf{H}^0 = \mathbf{B}$ and $\mathbf{S}^1 = \lambda f.\, \lambda\, \pi.\, \mathbf{S}({}^\vee \mathbf{f},\ {}^\vee \pi)$

(*c*) $\mathbf{H}_i{}^0(\mathbf{n}, \lambda i.\ \mathbf{S}_i{}^1(\lambda i.\, \mathbf{s}, \lambda i.\, \lambda i.\, \forall(\mathbf{B}_i)(\mathbf{D}_i)))$

$\equiv \mathbf{H}_i{}^0(\mathbf{n}, \lambda j.\, \mathbf{S}_j{}^1(\lambda j.\, \mathbf{s}, \lambda j.\, \lambda k.\, \forall(\mathbf{B}_k)(\mathbf{D}_k)))$

$\equiv \mathbf{H}_i^{0}(\mathbf{n}, \lambda j.\, \mathbf{S}_j^{1}(\lambda j.\, \mathbf{s}, \lambda j.\, \lambda k.\, \boxed{[\lambda j.\, \lambda k.\, \forall(\mathbf{B}_k)]}\ (j)(k)\ (\boxed{[\lambda j.\, \lambda k.\, \mathbf{D}_k]}(j)(k))))$ rigid in-intensions

(*d*) $\mathbf{H}_i^{0}(\mathbf{n}, \lambda j.\, \mathbf{S}_j^{1}(\lambda j.\, \mathbf{s}, \lambda j.\, \lambda k.\, \boxed{[\lambda j.\, \lambda k.\, \forall(\mathbf{B}_j)]}\ (j)(k)\ ([\lambda j.\, \lambda k.\, \mathbf{D}_k](j)(k))))$ twisted sense

$\equiv \mathbf{H}_i{}^0(\mathbf{n}, \lambda j.\, \mathbf{S}_j{}^1(\mathbf{s}, \lambda j.\, \lambda k.\, \forall(\mathbf{B}_j)(\mathbf{D}_k)))$

From rigid intensions …

$$[\![{}^{\wedge^n} \alpha]\!]^{w_0}(w_1)\ldots(w_n) = [\![\alpha]\!]^{w_n}$$

… to twisted senses:

$$[\![{}^{\wedge^n_m} \alpha]\!]^{w_0, g}(w_1)\ldots(w_n) = [\![\alpha]\!]^{w_m, g}$$

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Appendix: IL-implementation

a) Syntax of fragment

The fragment contains the key examples in the main text. The lexicon contains the following sets of expressions:

- *Names*: Δ_{Emily}, Δ_{Norman} , Δ_{Syd} ,…
- *Predicates* = Δ_{plays}, $\Delta_{is\ drinking}$ …
- *Attitude Verbs*: Δ_{sees}, Δ_{hears}, …
- *Nouns*: $\Delta_{band\ member}$, …
- *Determiners*: Δ_{every}, Δ_{a} , …

The syntax covers the constructions discussed above and contains the following rules:

R1 If Δ_{NN} is a *Name* and Δ_P is a predicate, then $\mathfrak{P}(\Delta_{NN}, \Delta_P)$ is a *Sentence*.

R2 If Δ_A is an *Attitude Verb* and Δ_S is a *Sentence*, then $\mathfrak{A}(\Delta_A, \Delta_S)$ is a *Predicate*.

R3 If Δ_D is a *Determiner* and Δ_N is a *Noun*, then $\mathfrak{D}(\Delta_D, \Delta_N)$ is a *Quantifier*.

R4 If Δ_Q is a *Quantifier* and Δ_P is a *Predicate*, then $\mathfrak{Q}(\Delta_Q, \Delta_P)$ is a *Sentence*.

b) *IL*: definitions and notation

(i) *Basic concepts*

The interpretation of the fragment will proceed indirectly, by way of a compositional interpretation into Montague's (1970) language *IL* of intensional type logic. The language is based on infinite sets Var_a of variables of any type *a* and unspecified sets Con_a of (non-logical) constants of type *a*, and consists of a set IL_a of terms of (any) type *a*:

- $Var_a \subseteq IL_a$.
- $Con_a \subseteq IL_a$.
- If $\alpha \in IL_{ab}$ and $\beta \in IL_a$, then $\alpha(\beta) \in IL_b$.
- If $\boldsymbol{x} \in IL_a$ and $\alpha \in IL_b$, then $(\boldsymbol{\lambda x.}\ \alpha) \in IL_{ab}$.
- If $\alpha \in IL_a$ and $\beta \in IL_a$, then $(\alpha \mathbf{=} \beta) \in IL_{\boldsymbol{t}}$.
- If $\alpha \in IL_{sa}$, then $[{}^{\vee}\alpha] \in IL_a$.
- If $\alpha \in IL_a$, then $[{}^{\wedge}\alpha] \in IL_{sa}$.

Following Montague (1970), logical constants and operators (like $\vee$, $\wedge$, $\exists$ and $\forall$) may be taken as abbreviations. *IL*-terms receive their denotations relative to models $\mathcal{M} = (D_{\boldsymbol{e}}, D_{\boldsymbol{s}}, \mathbf{F})$, indices $i \in D_{\boldsymbol{s}}$, and $\mathcal{M}$-assignments *g*:

- $[\![\boldsymbol{x}]\!]^{\mathcal{M},i,g} = g(\boldsymbol{x})$ if $\boldsymbol{x} \in Var_a$.
- $[\![\mathbf{c}]\!]^{\mathcal{M},i,g} = \mathbf{F}(\mathbf{c})(i)$ if $\mathbf{c} \in Con_a$.
- $[\![\alpha(\beta)]\!]^{\mathcal{M},i,g} = [\![\alpha]\!]^{\mathcal{M},i,g}([\![\beta]\!]^{\mathcal{M},i,g})$.
- $[\![(\lambda \boldsymbol{x}^a.\ \alpha)]\!]^{\mathcal{M},i,g} = \{(\mathrm{u}, [\![\alpha]\!]^{\mathcal{M},i,g[\boldsymbol{x}/u]}) \mid \mathrm{u} \in D_a\}$.
- $[\![(\alpha = \beta)]\!]^{\mathcal{M},i,g} = \{1 \mid [\![\alpha]\!]^{\mathcal{M},i,g} = [\![\beta]\!]^{\mathcal{M},i,g}\}$.
- $[\![[{}^{\vee}\alpha]]\!]^{\mathcal{M},i,g} = [\![\alpha]\!]^{\mathcal{M},i,g}(i)$.
- $[\![[{}^{\wedge}\alpha]]\!]^{\mathcal{M},i,g} = \{(j, [\![\alpha]\!]^{\mathcal{M},j,g}) \mid j \in D_s\}$.

Two *IL*-terms α and β of the same type are *logically equivalent* iff $[\![\alpha]\!]^{\mathcal{M},i,g} = [\![\beta]\!]^{\mathcal{M},i,g}$, for all models $\mathcal{M}$, indices i, and assignments g; notation: $\alpha \equiv \beta$.

(ii) *Iteration of IL-operators*

The indirect intensions in the hierarchy (29) are of the types of the form (s^na):

- $(\boldsymbol{s}^0 a) = a$
- $(\boldsymbol{s}^{n+1} a) = (\boldsymbol{s}(\boldsymbol{s}^n a))$

For each *IL*-term α the term $[{}^{\wedge^n}\alpha]$ denotes its n^{th} indirect intension:

- $[{}^{\wedge^0}\alpha] = \alpha$
- $[{}^{\wedge^{n+1}}\alpha] = [{}^{\wedge}[{}^{\wedge^n}\alpha]]$ $\quad (= [{}^{\wedge^n\wedge}\alpha])$

The indirect interpretation algorithms will also make use of iterated index application $[{}^{\vee^n}\alpha]$:

- $[{}^{\vee^{n+1}}\alpha] = [{}^{\vee^0}\alpha] = \alpha$
- $[{}^{\vee^{n+1}}\alpha] = [{}^{\vee}[{}^{\vee^n}\alpha]]$ $\quad (= [{}^{\vee^n\vee}\alpha])$

For each *IL*-term α, any $n\geq 0$ and $m\leq n$, the term $[{}^{\wedge^n_m}\alpha]$ designates the m^{th} twisted version of α's n^{th} indirect intension:

- $^{\wedge^n_m}\alpha = [{}^{\wedge^m}(\lambda \boldsymbol{X}.\, [{}^{\wedge^{n-m}}\boldsymbol{X}])(\alpha)]$ $\quad 0\leq m\leq n$

Functional application is defined recursively on the hierarchy of indirect intensions and twisted senses:

- $\mathbf{A}^0_{ab} = (\lambda \boldsymbol{f}.\, \lambda \boldsymbol{x}.\, \boldsymbol{f}(\boldsymbol{x}))$ $\quad \boldsymbol{f} \in Var_{(ab)}, \boldsymbol{x} \in Var_a$
- $\mathbf{A}^{n+1}_{ab} = (\lambda \boldsymbol{f}.\, \lambda \boldsymbol{x}.\, [{}^{\wedge}\mathbf{A}^n_{ab}([{}^{\vee}\boldsymbol{f}])([{}^{\vee}\boldsymbol{x}])])$ $\quad \boldsymbol{f} \in Var_{(s^{n+1}(ab))}, \boldsymbol{x} \in Var_{(s^{n+1}a)}$

c) *Indirect interpretation*

(i) *Standard translation*

For each expression Δ from the fragment defined in ***a)*** the *IL*-term $|\alpha|$ denotes its extension:

- $\{|\Delta_{Emily}|, |\Delta_{Norman}|, |\Delta_{Syd}|, \ldots\} = \{\mathbf{e}, \mathbf{n}, \mathbf{s}, \ldots\} \subseteq Con_{\boldsymbol{e}}$
- $\{|\Delta_{plays}|, \Delta_{is\ drinking}|, \ldots\} = \{\mathbf{P}, \mathbf{D}, \ldots\} \subseteq Con_{(\boldsymbol{et})}$
- $\{|\Delta_{sees}|, |\Delta_{hears}|, \ldots\} = \{\mathbf{S}, \mathbf{H}, \ldots\} \subseteq Con_{((\boldsymbol{st})\ \boldsymbol{et}))}$
- $\{|\Delta_{band\ member}|, \ldots\} = \{\mathbf{B}, \ldots\} \subseteq Con_{(\boldsymbol{et})}$
- $|\Delta_{every}| = [\lambda \boldsymbol{P}^{e\,t}.\, \lambda \boldsymbol{Q}^{e\,t}.\, (\forall \boldsymbol{x}^e)\, [\boldsymbol{P}(\boldsymbol{x}) \rightarrow \boldsymbol{Q}(\boldsymbol{x})]$ $\quad$ =: **ALL**
- $|\Delta_a| = [\lambda \boldsymbol{P}^{e\,t}.\, \lambda \boldsymbol{Q}^{e\,t}.\, (\exists \boldsymbol{x}^e)\, [\boldsymbol{P}(\boldsymbol{x})\,) \wedge \boldsymbol{Q}(\boldsymbol{x})]$

S1 $\quad |\mathfrak{P}(\Delta_{NN}, \Delta_P)| = |\Delta_P|\,(|\Delta_{NN}|)$

S2 $\quad |\mathfrak{A}(\Delta_A, \Delta_S)| = |\Delta_A|\,([{}^{\wedge}|\Delta_S|])$

S3 $\quad |\mathfrak{D}(\Delta_D, \Delta_N)| = |\Delta_D|\,(|\Delta_N|)$

S4 $\quad |\mathfrak{Q}(\Delta_Q, \Delta_P)| = |\Delta_Q|\,(|\Delta_P|)$

(ii) *Baroque translation*

For each expression Δ from the fragment defined in ***a)*** the *IL*-term $|\alpha|^n$ denotes its extension n^{th} indirect intension, which coincides with its extension if $n = 0$. In opaque positions the translation increases the level of indirectness :

- $|\Delta|^n = {}^{\wedge^n}|\Delta|$ if Δ is lexical

B1 $|\mathcal{P}(\Delta_{NN}, \Delta_P)|^n = \mathbf{A}^n_{\boldsymbol{et}} (|\Delta_P|^n)(|\Delta_{NN}|^n)$

B2 $|\mathcal{A}(\Delta_A, \Delta_S)|^n = \mathbf{A}^n_{\boldsymbol{(st)(et)}} (|\Delta_A|^n)(|\Delta_S|^{n+1})$

B3 $|\mathcal{D}(\Delta_D, \Delta_N)|^n = \mathbf{A}^n_{\boldsymbol{(et)((et)t)}} (|\Delta_D|^n)(|\Delta_N|^n)$

B4 $|\mathcal{Q}(\Delta_Q, \Delta_P)|^n = \mathbf{A}^n_{\boldsymbol{(et)t}} (|\Delta_Q|^n)(|\Delta_P|^n)$

(iii) *Underspecified translation*

For each expression Δ from the fragment defined in ***a)*** the set $|\alpha|^n_\sim$ of *IL*-terms contains all possible choices of n^{th} indirect twisted senses in flexible argument positions; in inflexible positions the semantic operations are distributed, flexible positions bring in twisted senses. The technique is the same as in Rooth's (1985) alternative semantics of focus:

- $|\Delta|^n_\sim = \{|\Delta|^n\}$, if Δ is lexical

U1 $|\mathcal{P}(\Delta_n, \Delta_P)|^n_\sim = \{\, \mathbf{A}^n_{\boldsymbol{et}}(\alpha)(\beta) \mid \alpha \in |\Delta_P|^n_\sim, \beta \in |\Delta_n|^n_\sim\}$

U2 $|\mathcal{A}(\Delta_A, \Delta_S)|^n_\sim = \{\, \mathbf{A}^n_{\boldsymbol{(st)(et)}}(\alpha)(\beta) \mid \alpha \in |\Delta_A|^n_\sim, \beta \in |\Delta_S|^{n+1}_\sim\}$

U3 $|\mathcal{D}(\Delta_D, \Delta_N)|^n_\sim = \{\mathbf{A}^n_{\boldsymbol{(et)((et)t)}}(\alpha)({}^{\wedge}{}^n_m{}^{\vee}{}^n\beta) \mid m \leq n, \alpha \in |\Delta_D|^n_\sim, \beta \in |\Delta_N|^n_\sim\}$

U4 $|\mathcal{Q}(\Delta_Q, \Delta_P)|^n_\sim = \{\, \mathbf{A}^n_{\boldsymbol{(et)t)}}(\alpha)(\beta) \mid \alpha \in |\Delta_Q|^n_\sim, \beta \in |\Delta_P|^n_\sim\}$

d) *Comparison*

(i) *Principal observations*

Let Δ be any expression in the above fragment. Then:

(P1) $|\Delta|^0 \equiv |\Delta|$

(P2) $|\Delta| \equiv \alpha \in |\Delta|^0_\sim$, for some *IL*-term α

(ii) *Auxiliary observations*

For all *IL*-terms α, $n \geq m \geq 0$, *IL*-models $\mathcal{M} = (D_{\boldsymbol{e}}, D_{\boldsymbol{s}}, \mathbf{F})$, $i_0,\ldots,i_{n+1} \in D_{\boldsymbol{s}}$, and $\mathcal{M}$-assignments g the following hold:

(A1) $[\![{}^{\vee n}{}^{\wedge n}\alpha]\!]^{\mathcal{M}, i_0, g} = [\![\alpha]\!]^{\mathcal{M}, i_0, g}$

(A2) $[\![{}^{\wedge n}\alpha]\!]^{\mathcal{M}, i_0, g}(i_1)\ldots(i_n) = [\![\alpha]\!]^{\mathcal{M}, i_n, g}$

(A3) $[\![{}^{\wedge n+1}\alpha]\!]^{\mathcal{M}, i_0, g}(i_1)\ldots(i_n) = [\![{}^{\wedge}\alpha]\!]^{\mathcal{M}, i_{n+1}, g}$

(A4) $[\![\mathbf{A}^n_{ab}({}^{\wedge n}\alpha)({}^{\wedge n}\beta)]\!]^{\mathcal{M}, i_0, g} = [\![{}^{\wedge n}\alpha(\beta)]\!]^{\mathcal{M}, i_0, g}$ where, for some types a and b, $\alpha \in IL_{(s^n(ab))}$, $\beta \in IL_{(s^n a)}$

(A5) $|\Delta|^n \equiv [{}^{\wedge n}|\Delta|]$ ⇒ *(P1)*

(A6) $[\![{}^{\wedge}{}^n_m\alpha]\!]^{\mathcal{M}, i_0, g}(i_1)\ldots(i_n) = [\![\alpha]\!]^{\mathcal{M}, i_m, g}$

(A7) $[\![{}^{\wedge}{}^n_n\alpha]\!]^{\mathcal{M}, i_0, g} = [\![{}^{\wedge n}\alpha]\!]^{\mathcal{M}, i_0, g}$

(A8) $[\![\mathbf{A}^{n}_{ab}(\alpha)(\beta)]\!]^{\mathcal{M},i_0,g}(i_1)\ldots(i_n) = [\![\alpha]\!]^{\mathcal{M},i_0,g}(i_1)\ldots(i_n)\,([\![\beta]\!]^{\mathcal{M},i_0,g}(i_1)\ldots(i_n))$

where α and β are as in *(A4)*

(A9) $[\wedge^{n}|\Delta|\,] \equiv \alpha \in |\,\Delta\,|^{n}_{\sim}$, for some *IL*-term α $\Rightarrow$ *(P2)*

e) *Example*

(36) Norman hears that Syd sees that every band member is drinking.

- *Underlying structure:*

$\mathcal{P}(\Delta_{Norman}, \mathcal{A}(\Delta_{hears}, \mathcal{P}(\Delta_{Syd}, \mathcal{A}(\Delta_{sees}, \mathcal{Q}(\mathcal{D}(\Delta_{every}, \Delta_{band\ member}), \Delta_{is\ drinking})))))$

(i) *Standard translation*

- $|\,\mathcal{Q}(\mathcal{D}(\Delta_{every}, \Delta_{band\ member}), \Delta_{is\ drinking})\,|$

$=$ $|\,\Delta_{every}\,|(|\,\Delta_{band\ member}\,|)\,(|\,\Delta_{is\ drinking}\,|)$

$=$ $\mathbf{ALL(B)\ (D)}$

$\equiv$ $(\forall \boldsymbol{x}^{e})\,[\mathbf{B}(\boldsymbol{x}) \rightarrow \mathbf{D}(\boldsymbol{x})]$

- $|\,\mathcal{P}(\Delta_{Syd}, \mathcal{A}(\Delta_{sees}, \mathcal{Q}(\mathcal{D}(\Delta_{every}, \Delta_{band\ member}), \Delta_{is\ drinking})))\,|$

$=$ $\mathbf{S(s,}\,[\wedge\,|\,\mathcal{Q}(\mathcal{D}(\Delta_{every}, \Delta_{band\ member}), \Delta_{is\ drinking})\,|\,]\mathbf{)}$

$\equiv$ $\mathbf{S(s,}\,[\wedge\,(\forall \boldsymbol{x}^{e})\,[\mathbf{B}(\boldsymbol{x}) \rightarrow \mathbf{D}(\boldsymbol{x})]\,]\mathbf{)}$

- $|\,(36)\,|$

$=$ $\mathbf{H(n,}\,[\wedge\,|\,\mathcal{P}(\Delta_{Syd}, \mathcal{A}(\Delta_{sees}, \mathcal{Q}(\mathcal{D}(\Delta_{every}, \Delta_{band\ member}), \Delta_{is\ drinking})))\,|\,]\mathbf{)}$

$\equiv$ $\mathbf{H(n,}\,[\wedge\mathbf{S(s,}\,[\wedge\,(\forall \boldsymbol{x}^{e})\,[\mathbf{B}(\boldsymbol{x}) \rightarrow \mathbf{D}(\boldsymbol{x})]\,]\,]\mathbf{)}$

$\Rightarrow$ $[\![\,|\,(36)\,|\,]\!]^{\mathcal{M},i_0,g} = H_i(\mathrm{n}, \lambda j.\, S_j(\lambda k.\, T_k \subseteq D_k))(\mathrm{s}))$ using notational conventions from the text

(ii) *Baroque translation*

- $|\,\mathcal{P}(\Delta_{Norman}, \mathcal{A}(\Delta_{hears}, \mathcal{P}(\Delta_{Syd}, \mathcal{A}(\Delta_{sees}, \mathcal{Q}(\mathcal{D}(\Delta_{every}, \Delta_{team\ member}), \Delta_{is\ drinking})))))\,|^{0}$

$=$ $[\![\mathbf{A}^{0}_{et}(\mathbf{A}^{0}_{(st)(et)}(\mathbf{H},\mathbf{A}^{1}_{et}(\mathbf{A}^{1}_{(st)(et)}(^\wedge\mathbf{S},\mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}(^{\wedge\wedge}\mathbf{ALL},^{\wedge\wedge}\mathbf{T}),^{\wedge\wedge}\mathbf{D})),^{\wedge}\mathbf{s})),\mathbf{n})]\!]^{\mathcal{M},i,g}$

$=$ $[\![\mathbf{H}]\!]^{\mathcal{M},i,g}([\![\mathbf{n}]\!]^{\mathcal{M},i,g}, \lambda j.\, [\![\mathbf{S}]\!]^{\mathcal{M},j,g}(\lambda k.\, [\![^{\wedge\wedge}\mathbf{ALL}]\!]^{\mathcal{M},i,g}(j)(k)([\![^{\wedge\wedge}\mathbf{T}]\!]^{\mathcal{M},i,g}(j)(k))([\![^{\wedge\wedge}\mathbf{D}]\!]^{\mathcal{M},j,g})(j)(k))([\![\mathbf{s}]\!]^{\mathcal{M},j,g}))$

$=$ $[\![\mathbf{H}]\!]^{\mathcal{M},i,g}([\![\mathbf{n}]\!]^{\mathcal{M},i,g}, \lambda j.\, [\![\mathbf{S}]\!]^{\mathcal{M},j,g}(\lambda k.\, [\![\mathbf{ALL}]\!]^{\mathcal{M},k,g}([\![\mathbf{T}]\!]^{\mathcal{M},k,g})([\![\mathbf{D}]\!]^{\mathcal{M},k,g}))([\![\mathbf{s}]\!]^{\mathcal{M},j,g}))$

$=$ $H_i(\mathrm{n}, \lambda j.\, S_j(\lambda k.\, T_k \subseteq D_k))(\mathrm{s}))$ $= [\![\,|\,(36)\,|\,]\!]^{\mathcal{M},i_0,g}$

(iii) *Underspecified translation*

- $|\,\mathcal{P}(\Delta_{Norman}, \mathcal{A}(\Delta_{hears}, \mathcal{P}(\Delta_{Syd}, \mathcal{A}(\Delta_{sees}, \mathcal{Q}(\mathcal{D}(\Delta_{every}, \Delta_{team\ member}), \Delta_{is\ drinking})))))\,|^{0}_{\sim}$

$=$ $\{\mathbf{A}^{0}_{et}(\mathbf{A}^{0}_{(st)(et)}(\alpha,\mathbf{A}^{1}_{et}(\mathbf{A}^{1}_{(st)(et)}(\beta,\mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}(\gamma,{}^{\wedge}{}^{2}_{m}{}^{\vee}{}^{2}\delta),\varepsilon)),\sigma)),\nu)\mid m \leq 2,$
$\alpha \in |\,\Delta_{hears}\,|^{0}_{\sim}, \alpha \in |\,\Delta_{sees}\,|^{1}_{\sim}, \gamma \in |\,\Delta_{every}\,|^{2}_{\sim}, \delta \in |\,\Delta_{team\ member}\,|^{2}_{\sim},$
$\varepsilon \in |\,\Delta_{is\ drinking}\,|^{2}_{\sim}, \sigma \in |\,\Delta_{Syd}\,|^{1}_{\sim}, \nu \in |\,\Delta_{Norman}\,|^{0}_{\sim}\}$

$=$ $\{\mathbf{A}^{0}_{et}(\mathbf{A}^{0}_{(st)(et)}(\mathbf{H},\mathbf{A}^{1}_{et}(\mathbf{A}^{1}_{(st)(et)}(^\wedge\mathbf{S},\mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}(^{\wedge\wedge}\mathbf{ALL},{}^{\wedge}{}^{2}_{0}{}^{\vee}{}^{2}\,^{\wedge\wedge}\mathbf{T}),^{\wedge\wedge}\mathbf{D})),^{\wedge}\mathbf{s})),\mathbf{n}),$
$\mathbf{A}^{0}_{et}(\mathbf{A}^{0}_{(st)(et)}(\mathbf{H},\mathbf{A}^{1}_{et}(\mathbf{A}^{1}_{(st)(et)}(^\wedge\mathbf{S},\mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}(^{\wedge\wedge}\mathbf{ALL},{}^{\wedge}{}^{2}_{1}{}^{\vee}{}^{2}\,^{\wedge\wedge}\mathbf{T}),^{\wedge\wedge}\mathbf{D})),^{\wedge}\mathbf{s})),\mathbf{n}),$

$$\mathbf{A}^{0}_{et}(\mathbf{A}^{0}_{(st)(et)}(\mathbf{H},\mathbf{A}^{1}_{et}(\mathbf{A}^{1}_{(st)(et)}({}^\wedge\mathbf{S},\mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}({}^{\wedge\wedge}\mathbf{ALL},{}^\wedge{}^{2}_{2}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T}),{}^{\wedge\wedge}\mathbf{D})),{}^\wedge\mathbf{s})),\mathbf{n})\}$$

$$\bullet \quad \left\llbracket \mathbf{A}^{0}_{et}(\mathbf{A}^{0}_{(st)(et)}(\mathbf{H},\mathbf{A}^{1}_{et}(\mathbf{A}^{1}_{(st)(et)}({}^\wedge\mathbf{S},\mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}({}^{\wedge\wedge}\mathbf{ALL},{}^\wedge{}^{2}_{0}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T}),{}^{\wedge\wedge}\mathbf{D})),{}^\wedge\mathbf{s})),\mathbf{n})\right\rrbracket^{\mathbb{M},i,g}$$

$$\overset{A8}{=} \quad \left\llbracket \mathbf{A}^{0}_{(st)(et)}(\mathbf{H},\mathbf{A}^{1}_{et}(\mathbf{A}^{1}_{(st)(et)}({}^\wedge\mathbf{S},\mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}({}^{\wedge\wedge}\mathbf{ALL},{}^\wedge{}^{2}_{0}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T}),{}^{\wedge\wedge}\mathbf{D})),{}^\wedge\mathbf{s}))\right\rrbracket^{\mathbb{M},i,g}(\llbracket\mathbf{n}\rrbracket^{\mathbb{M},i,g})$$

$$\overset{A8}{=} \quad \llbracket\mathbf{H}\rrbracket^{\mathbb{M},i,g}(\llbracket\mathbf{n}\rrbracket^{\mathbb{M},i,g},\left\llbracket \mathbf{A}^{1}_{et}(\mathbf{A}^{1}_{(st)(et)}({}^\wedge\mathbf{S},\mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}({}^{\wedge\wedge}\mathbf{ALL},{}^\wedge{}^{2}_{0}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T}),{}^{\wedge\wedge}\mathbf{D})),{}^\wedge\mathbf{s})\right\rrbracket^{\mathbb{M},i,g})$$

$$\overset{\eta\text{ conv.}}{=} \quad \llbracket\mathbf{H}\rrbracket^{\mathbb{M},i,g}(\llbracket\mathbf{n}\rrbracket^{\mathbb{M},i,g},\lambda j.\left\llbracket \mathbf{A}^{1}_{et}(\mathbf{A}^{1}_{(st)(et)}({}^\wedge\mathbf{S},\mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}({}^{\wedge\wedge}\mathbf{ALL},{}^\wedge{}^{2}_{0}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T}),{}^{\wedge\wedge}\mathbf{D})),{}^\wedge\mathbf{s})\right\rrbracket^{\mathbb{M},i,g}(j))$$

$$\overset{A8}{=} \quad \llbracket\mathbf{H}\rrbracket^{\mathbb{M},i,g}(\llbracket\mathbf{n}\rrbracket^{\mathbb{M},i,g},\lambda j.\left\llbracket \mathbf{A}^{1}_{(st)(et)}({}^\wedge\mathbf{S},\mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}({}^{\wedge\wedge}\mathbf{ALL},{}^\wedge{}^{2}_{0}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T}),{}^{\wedge\wedge}\mathbf{D}))\right\rrbracket^{\mathbb{M},i,g}(j)\,(\llbracket{}^\wedge\mathbf{s}\rrbracket^{\mathbb{M},i,g}(j)))$$

$$\overset{IL}{=} \quad \llbracket\mathbf{H}\rrbracket^{\mathbb{M},i,g}(\llbracket\mathbf{n}\rrbracket^{\mathbb{M},i,g},\lambda j.\left\llbracket \mathbf{A}^{1}_{(st)(et)}({}^\wedge\mathbf{S},\mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}({}^{\wedge\wedge}\mathbf{ALL},{}^\wedge{}^{2}_{0}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T}),{}^{\wedge\wedge}\mathbf{D}))\right\rrbracket^{\mathbb{M},i,g}(j)\,(\llbracket\mathbf{s}\rrbracket^{\mathbb{M},j,g}))$$

$$\overset{A8}{=} \quad \llbracket\mathbf{H}\rrbracket^{\mathbb{M},i,g}(\llbracket\mathbf{n}\rrbracket^{\mathbb{M},i,g},\lambda j.\,\llbracket{}^\wedge\mathbf{S}\rrbracket^{\mathbb{M},i,g}(j)(\left\llbracket \mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}({}^{\wedge\wedge}\mathbf{ALL},{}^\wedge{}^{2}_{0}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T}),{}^{\wedge\wedge}\mathbf{D})\right\rrbracket^{\mathbb{M},i,g}(j))(\llbracket\mathbf{s}\rrbracket^{\mathbb{M},j,g}))$$

$$\overset{IL}{=} \quad \llbracket\mathbf{H}\rrbracket^{\mathbb{M},i,g}(\llbracket\mathbf{n}\rrbracket^{\mathbb{M},i,g},\lambda j.\,\llbracket\mathbf{S}\rrbracket^{\mathbb{M},j,g}(\left\llbracket \mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}({}^{\wedge\wedge}\mathbf{ALL},{}^\wedge{}^{2}_{0}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T}),{}^{\wedge\wedge}\mathbf{D})\right\rrbracket^{\mathbb{M},i,g}(j))(\llbracket\mathbf{s}\rrbracket^{\mathbb{M},j,g}))$$

$$\overset{\eta\text{ conv.}}{=} \quad \llbracket\mathbf{H}\rrbracket^{\mathbb{M},i,g}(\llbracket\mathbf{n}\rrbracket^{\mathbb{M},i,g},\lambda j.\,\llbracket\mathbf{S}\rrbracket^{\mathbb{M},j,g}(\lambda k.\,\left\llbracket \mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}({}^{\wedge\wedge}\mathbf{ALL},{}^\wedge{}^{2}_{0}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T}),{}^{\wedge\wedge}\mathbf{D})\right\rrbracket^{\mathbb{M},i,g}(j)(k))(\llbracket\mathbf{s}\rrbracket^{\mathbb{M},j,g}))$$

$$\overset{A8}{=} \quad \llbracket\mathbf{H}\rrbracket^{\mathbb{M},i,g}(\llbracket\mathbf{n}\rrbracket^{\mathbb{M},i,g},\lambda j.\,\llbracket\mathbf{S}\rrbracket^{\mathbb{M},j,g}(\lambda k.\,\left\llbracket \mathbf{A}^{2}_{(et)((et)t)}({}^{\wedge\wedge}\mathbf{ALL},{}^\wedge{}^{2}_{0}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T})\right\rrbracket^{\mathbb{M},i,g}(j)(k)(\llbracket{}^{\wedge\wedge}\mathbf{D}\rrbracket^{\mathbb{M},i,g})(j)(k))(\llbracket\mathbf{s}\rrbracket^{\mathbb{M},j,g}))$$

$$\overset{A8}{=} \quad \llbracket\mathbf{H}\rrbracket^{\mathbb{M},i,g}(\llbracket\mathbf{n}\rrbracket^{\mathbb{M},i,g},\lambda j.\,\llbracket\mathbf{S}\rrbracket^{\mathbb{M},j,g}(\lambda k.\,\llbracket{}^{\wedge\wedge}\mathbf{ALL}\rrbracket^{\mathbb{M},i,g}(j)(k)(\left\llbracket{}^\wedge{}^{2}_{0}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T}\right\rrbracket^{\mathbb{M},i,g}(j)(k))(\llbracket{}^{\wedge\wedge}\mathbf{D}\rrbracket^{\mathbb{M},j,g})(j)(k))(\llbracket\mathbf{s}\rrbracket^{\mathbb{M},j,g}))$$

$$\overset{A1}{=} \quad \llbracket\mathbf{H}\rrbracket^{\mathbb{M},i,g}(\llbracket\mathbf{n}\rrbracket^{\mathbb{M},i,g},\lambda j.\,\llbracket\mathbf{S}\rrbracket^{\mathbb{M},j,g}(\lambda k.\,\llbracket\mathbf{ALL}\rrbracket^{\mathbb{M},k,g}(\left\llbracket{}^\wedge{}^{2}_{0}\mathbf{T}\right\rrbracket^{\mathbb{M},i,g}(j)(k))(\llbracket\mathbf{D}\rrbracket^{\mathbb{M},k,g}))(\llbracket\mathbf{s}\rrbracket^{\mathbb{M},j,g}))$$

$$\overset{A6}{=} \quad \llbracket\mathbf{H}\rrbracket^{\mathbb{M},i,g}(\llbracket\mathbf{n}\rrbracket^{\mathbb{M},i,g},\lambda j.\,\llbracket\mathbf{S}\rrbracket^{\mathbb{M},j,g}(\lambda k.\,\llbracket\mathbf{ALL}\rrbracket^{\mathbb{M},k,g}(\llbracket\mathbf{T}\rrbracket^{\mathbb{M},i,g})(\llbracket\mathbf{D}\rrbracket^{\mathbb{M},k,g}))(\llbracket\mathbf{s}\rrbracket^{\mathbb{M},j,g}))$$

$$\overset{not.}{=} \quad H_i(\mathrm{n},\lambda j.\,S_j(\lambda k.\,T_i\subseteq D_k))(\mathrm{s}))$$

$$\bullet \quad \left\llbracket \mathbf{A}^{0}_{et}(\mathbf{A}^{0}_{(st)(et)}(\mathbf{H},\mathbf{A}^{1}_{et}(\mathbf{A}^{1}_{(st)(et)}({}^\wedge\mathbf{S},\mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}({}^{\wedge\wedge}\mathbf{ALL},{}^\wedge{}^{2}_{1}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T}),{}^{\wedge\wedge}\mathbf{D})),{}^\wedge\mathbf{s})),\mathbf{n})\right\rrbracket^{\mathbb{M},i,g}$$

$$= \quad \dots$$

$$= \quad H_i(\mathrm{n},\lambda j.\,S_j(\lambda k.\,T_j\subseteq D_k))(\mathrm{s}))$$

$$\bullet \quad \left\llbracket \mathbf{A}^{0}_{et}(\mathbf{A}^{0}_{(st)(et)}(\mathbf{H},\mathbf{A}^{1}_{et}(\mathbf{A}^{1}_{(st)(et)}({}^\wedge\mathbf{S},\mathbf{A}^{2}_{((et)t}(\mathbf{A}^{2}_{(et)((et)t)}({}^{\wedge\wedge}\mathbf{ALL},{}^\wedge{}^{2}_{2}{\vee}^{2}{}^{\wedge\wedge}\mathbf{T}),{}^{\wedge\wedge}\mathbf{D})),{}^\wedge\mathbf{s})),\mathbf{n})\right\rrbracket^{\mathbb{M},i,g}$$

$$= \quad \dots$$

$$= \quad H_i(\mathrm{n},\lambda j.\,S_j(\lambda k.\,T_k\subseteq D_k))(\mathrm{s})) \qquad = \llbracket\,|\,(36)\,|\,\rrbracket^{\mathbb{M},i_0,g}$$