A Appendix: Proofs

A.1 Selected Observations

ad **h)**: $a \in \mathbf{RT}$ iff $a^* = a$.
We reason by induction on $|a|$, where $a \in \mathbf{FT}$. If $|a| = 1$, then $a \in \{\mathrm{t,e}\}$ and hence $a \in \mathbf{RT}$ and $a^* = a$ by definition. For the inductive step, we separate the directions. " $\Rightarrow$ ": There are two cases: either (a) $a = (\mathrm{st})$; or (b) $a = (bc)$, where $b, c \in \mathbf{RT}$. If (a), $a^* = a$, again by definition; if (b), $b = b^*$ and $c = c^*$, by induction. Moreover, $(bc)^* = (b^*c^*)$, by the definition of the shift * and given that $b = b^*$. Hence $a^* = (b^*c^*) = (bc)$. – " $\Leftarrow$ ": If $|a| > 1$ and $a \in \mathbf{FT}$, then there are two cases: either (c) $a = (bc)$, or (d) $a = (\mathrm{s}c)$, where $b, c \in \mathbf{FT}$; in both cases, we reason by contraposition. Turning to (c), an inspection of the definition of * reveals that $a^* = (b^*d)$, for some $d \in \mathbf{2T}$. However, if $a \notin \mathbf{RT}$, then either (c1) $b \notin \mathbf{RT}$ or else (c2) $c \notin \mathbf{RT}$, given the definition of $\mathbf{RT}$. But in case (c1), $b \neq b^*$, by induction, and thus $a^* = (b^*d) \neq (bc) = a$, by the characteristic property of ordered pairs.[i] Yet if (c2) applies, $c \neq c^*$, by induction, and at the same time, $d = c^*$ or $d = (c^*\mathrm{t})$, depending on whether c^* is Boolean. Either way, $d \neq c : c \neq c^*$ by assumption, and $c \neq (c^*\mathrm{t})$ because $|c^*\mathrm{t}| > |c|$, by **f)**. So, $a^* = (b^*d) \neq (bc) = a$. – As to case (d), the two basic sub-cases are immediate: $(\mathrm{se})^* = (\mathrm{e(st)}) \neq (\mathrm{se})$; and $(\mathrm{st}) \in \mathbf{RT}$, anyway. In the remaining two cases, $a^* = (bd)$, for certain $b, d \in \mathbf{FT}$ – and thus: $a^* = (bd) \neq (\mathrm{s}c)$, because $\mathrm{s} \notin \mathbf{FT}$.

ad **j)**: If $a \notin \mathbf{RT}$, $a^* \in \mathbf{BT}$.
An induction on $a \in \mathbf{FT}$ reveals that:

j*) If $a^* \notin \mathbf{BT}, a^* = a$,

which is equivalent to **j)**, in view of **h)**. More specifically, if $a \in \{\mathrm{e}, \mathrm{t}\}$, **j*)** holds trivially. If $a = (bc)$ and $a^* \notin \mathbf{BT}$, then $a^* = (b^*c^*)$, where $c^* \notin \mathbf{BT}$. But then $b = b^*$, by the definition of $(bc)^*$; moreover, by induction, $c^* = c$. So $a^* = (b^*c^*) = (bc) = a$. Finally, if $a = (\mathrm{s}b)$, $a^* \in \mathbf{BT}$, by **i)**.

ad **m)**: $\dot{\kappa}^{\cup}$ is (i) surjective but (ii) not injective.
It should be noted that $\dot{\kappa}^{\cup}$: $\bigcup_{a \in \mathbf{FT}} D_a \longrightarrow \bigcup_{a \in \mathbf{RT}} D_a$ only because we have been assuming that $(+)$ $D_a \cap D_b = \varnothing$ whenever $a \neq b$; otherwise $\dot{\kappa}^{\cup}$ might not even be a function to begin with. But given $(+)$, (i) immediately follows from **l)**: if $a \in \mathbf{RT}$ and $\boldsymbol{y} \in D_b$, then $\boldsymbol{y} = \boldsymbol{id}_b(\boldsymbol{y}) = \dot{\kappa}_b(\boldsymbol{y}) = \dot{\kappa}^{\cup}(\boldsymbol{y})$. As to (ii), whenever $a \in \mathbf{FT} \setminus \mathbf{RT}$ and $\boldsymbol{x} \in D_a$, $\dot{\kappa}_a(\boldsymbol{x}) = \boldsymbol{y} \in D_{a^*}$, where $a^* \in \mathbf{RT}$. Hence, by **l)**, $\dot{\kappa}_{a^*}(\boldsymbol{y}) = \boldsymbol{id}_{a^*}(\boldsymbol{y}) = \boldsymbol{y}$. But then: $\dot{\kappa}^{\cup}(\boldsymbol{x}) = \dot{\kappa}_a(\boldsymbol{x}) = \boldsymbol{y} = \dot{\kappa}_{a^*}(\boldsymbol{y}) = \dot{\kappa}^{\cup}(\boldsymbol{y})$, even though $\boldsymbol{x} \neq \boldsymbol{y}$, due to the disjointness assumption $(+)$.

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[i] However ordered pairs are defined, they satisfy the condition that $(x, y) = (x', y')$ iff $x = x'$ and $y = y'$.

ad **n)**: $g[{}^{x}/_{\boldsymbol{u}}]^* = g^*[{}^{\tilde{x}}/_{\dot{\kappa}_a(\boldsymbol{u})}]$
To begin with, we have:

$g[{}^{x}/_{\boldsymbol{u}}]^*(\tilde{x}) =$
$\dot{\kappa}_a(g[{}^{x}/_{\boldsymbol{u}}](x)) =$ by the definition of h^*
$\dot{\kappa}_a(\boldsymbol{u}) =$ by the definition of $h[{}^{y}/_{\boldsymbol{v}}]$
$g^*[{}^{\tilde{x}}/_{\dot{\kappa}_a(\boldsymbol{u})}](\tilde{x})$ by the same definition

But then, for any $y = z^* \in Var \setminus \{\tilde{x}, i\}$:

$g[{}^{x}/_{\boldsymbol{u}}]^*(y) =$
$\dot{\kappa}_a(g[{}^{x}/_{\boldsymbol{u}}](z)) =$ by the definition of h^*
$\dot{\kappa}_a(g(z)) =$ $y \neq \tilde{x}$ and thus $z \neq x$
$g^*(y) =$ by the definition of h^*
$g^*[{}^{\tilde{x}}/_{\dot{\kappa}_a(\boldsymbol{u})}](y)$ $y \neq \tilde{x}$

ad **q)**: The two claims being very similar, we only prove the second one: $\dot{\sqcap}_b(/\boldsymbol{X}/) = \boldsymbol{X}$.
Given $b = (\overrightarrow{b}\,\mathrm{t}) \in \mathbf{RT} \cap \mathbf{BT}$, where $\overrightarrow{b} = \langle b_1, \ldots, b, \mathrm{s}\rangle$, $\boldsymbol{X} \in D_b$ and arbitrary $\boldsymbol{x}_1 \in D_{b_1}, \ldots, \boldsymbol{x}_n \in D_{b_n}$ and $\boldsymbol{w} \in D_{\mathrm{s}}$, we have:

$\dot{\sqcap}_b(/\boldsymbol{X}/)(\boldsymbol{x}_1)\ldots(\boldsymbol{x}_n)(\boldsymbol{w}) = 1$
iff $\boldsymbol{P}(\boldsymbol{x}_1)\ldots(\boldsymbol{x}_n)(\boldsymbol{w}) = 1$, for any $\boldsymbol{P} \in D_b$ such that $/\boldsymbol{X}/(\boldsymbol{P}) = 1$ IL^*-int.
iff $\boldsymbol{P}(\boldsymbol{x}_1)\ldots(\boldsymbol{x}_n)(\boldsymbol{w}) = 1$, for any $\boldsymbol{P} \in D_b$ such that $\boldsymbol{P} = \boldsymbol{X}$ def. $/\cdot/$
iff $\boldsymbol{X}(\boldsymbol{x}_1)\ldots(\boldsymbol{x}_n)(\boldsymbol{w}) = 1$, identity

which means that $\dot{\sqcap}_b(/\boldsymbol{X}/) = \boldsymbol{X}$, by extensionality.

ad **t)**: $\dot{\kappa}_a$ is (aa^*)-invariant .
Let $(\pi_a)_{a\in\mathbf{2T}}$ be a permutation of $(D_a)_{a\in\mathbf{2T}}$ (where D_{e} and D_{s} are arbitrary, non-empty sets). Then for any IL^*-model F and IL^*-assignment g, the corresponding *permutation model* F_π and *permutation assignment* g_π are defined as: $F_\pi := \pi \circ F$ and $g_\pi := \pi \circ g$, where '$\circ$' denotes functional composition.[ii] A routine induction shows that, for any $a \in \mathbf{FT}$ and any $\alpha \in IL^*_a : (*)\ [\![\alpha]\!]^{F_\pi, g_\pi} = \pi([\![\alpha]\!]^{F,g})$. But then:

	$\dot{\kappa}_a$	
$=$	$[\![\kappa_a]\!]^{F,g}$	def. $\dot{\kappa}_a$
$=$	$[\![\kappa_a]\!]^{F_\pi, g_\pi}$	Coincidence Lemma[iii]
$=$	$\pi([\![\kappa_a]\!]^{F,g})$	(*)
$=$	$\pi(\dot{\kappa}_a)$	def. $\dot{\kappa}_a$

The logicality of aa^* and the embedability of a into a^* then immediately follow from the κ-Lemma and the pertinent definitions.

[ii] In other words, $F_\pi(\mathrm{c}) = \pi(F(\mathrm{c}))$ and $g_\pi(x) = \pi(g(x))$ for any $\mathrm{c} \in Con$ and $x \in Var$.
[iii] The very $\dot{\kappa}$-notation had already taken this step for granted. Strictly speaking, it requires a more general (and standard) version of the Coincidence Lemma that generalizes over models as well as sets of individuals and words.

ad **u)**: $((se)e)((se)e)^o = ((se)e)((e(st))e)$ is not logical.
To be sure, $((se)e)^o = (se)^o e^o = (e^o(st))e^o = ((e(st))e$, by the very definition of Kaplan's shift ^{o}. Taking advantage of the model-theoretic perspective, we assume that $D_s = \{\boldsymbol{w}_0\}$ is a singleton, whereas D_e has at least two members, and show that for any $\boldsymbol{F} \in D_{((se)e)((e(st))e)}$ there is a permutation π of $(D_a)_{a\in\mathbf{2T}}$ such that $\pi_{((se)e)((e(st))e)}(\boldsymbol{F}) \neq \boldsymbol{F}$. To adopt a strategem of van Benthem [1, p. 319], we first observe that, given the circumstances, the domains D_e and D_{se} are isomorphic in that the elements of D_{se} are of the form $\boldsymbol{c_x} := \{(\boldsymbol{w}_0, \boldsymbol{x})\}$, where $\boldsymbol{x} \in D_e$ and $\boldsymbol{c_x} \neq \boldsymbol{c_y}$ for distinct $\boldsymbol{x}$, $\boldsymbol{y} \in D_e$. Moreover, for any permutation π and any $\boldsymbol{c_x} \in D_{se}$ we have:

$(*)$	$\pi_{se}(\boldsymbol{c_x})(\boldsymbol{w}_0)$	
$=$	$\pi_e(\boldsymbol{c_x}(\pi_s^{-1}(\boldsymbol{w}_0)))$	def. π_{se}
$=$	$\pi_e(\boldsymbol{c_x}(\boldsymbol{w}_0))$	$D_s = \{\boldsymbol{w}_0\}$
$=$	$\pi_e(\boldsymbol{x})$	def. $\boldsymbol{c_x}$
$=$	$\boldsymbol{c}_{\pi_e(\boldsymbol{x})}(\boldsymbol{w}_0)$	def. $\boldsymbol{c}_{\pi_e(\boldsymbol{x})}$

In other words:

$\binom{*}{*}$ $\quad \pi_{se}(\boldsymbol{c_x}) = \boldsymbol{c}_{\pi_e(\boldsymbol{x})}$,

by extensionality and given that $D_s = \{\boldsymbol{w}_0\}$. As a consequence, the function $\boldsymbol{g} := \{(\boldsymbol{c_x}, \boldsymbol{x}) | \boldsymbol{x} \in D_e\}$ is ((se)e)-invariant; for if π is a permutation and $\boldsymbol{c_x} \in D_{se}$, we have:

	$\pi_{(se)e}(\boldsymbol{g})(\boldsymbol{c_x})$	
$=$	$\pi_e(\boldsymbol{g}(\pi_{se}^{-1}(\boldsymbol{c_x})))$	def. $\pi_{(se)e}$
$=$	$\pi_e(\boldsymbol{g}(\boldsymbol{c}_{\pi_e^{-1}(\boldsymbol{x})}))$	by $\binom{*}{*}$
$=$	$\pi_e(\pi_e^{-1}(\boldsymbol{x}))$	def. $\boldsymbol{g}$
$=$	$\boldsymbol{x}$	$\pi_e \circ \pi_e{-1} = \boldsymbol{id}_e$
$=$	$\boldsymbol{g}(\boldsymbol{c_x})$	def. $\boldsymbol{g}$

Now suppose $\boldsymbol{F} \in D_{((se)e)((e(st))e)}$ and put: $\boldsymbol{x}_0 := \boldsymbol{F}(\boldsymbol{g})(\boldsymbol{0}_{e(st)}) \neq \boldsymbol{x}_1 \in D_e$. We may then consider the transposition

$\pi^{\lightning} := [\boldsymbol{id}_e \setminus \{(\boldsymbol{x}_0, \boldsymbol{x}_0), (\boldsymbol{x}_1, \boldsymbol{x}_1)\}] \cup \{(\boldsymbol{x}_0, \boldsymbol{x}_1), (\boldsymbol{x}_1, \boldsymbol{x}_0)\}$

and observe:

	$\pi^{\lightning}_{((se)e)((e(st))e)}(\boldsymbol{F})(\boldsymbol{g})(\boldsymbol{0}_{e(st)})$	
$=$	$\pi^{\lightning}_{(e(st))e}(\boldsymbol{F}([\pi^{\lightning}_{(se)e}]^{-1}(\boldsymbol{g})))(\boldsymbol{0}_{e(st)})$	def. $\pi^{\lightning}_{((se)e)((e(st))e)}$
$=$	$\pi^{\lightning}_{(e(st))e}(\boldsymbol{F}(\boldsymbol{g}))(\boldsymbol{0}_{e(st)})$	$\boldsymbol{g}$'s invariance
$=$	$\pi^{\lightning}_e(\boldsymbol{F}(\boldsymbol{g})([\pi^{\lightning}_{e(st)}]^{-1}(\boldsymbol{0}_{e(st)})))$	def. $\pi^{\lightning}_{(e(st))e}$
$=$	$\pi^{\lightning}_e(\boldsymbol{F}(\boldsymbol{g})(\boldsymbol{0}_{e(st)}))$	$\boldsymbol{0}_{e(st)}$'s invariance)
$=$	$\pi^{\lightning}_e(\boldsymbol{x}_0)$	def. $\boldsymbol{x}_0$
$=$	$\boldsymbol{x}_1$	def. $\pi^{\lightning}_e$
$\neq$	$\boldsymbol{x}_0$	assumption
$=$	$\boldsymbol{F}(\boldsymbol{g})(\boldsymbol{0}_{e(st))})$	def. $\boldsymbol{x}_0$

Hence $\pi^{\lightning}_{((se)e)((e(st))e)}(\boldsymbol{F}) \neq \boldsymbol{F}$, which means that $\boldsymbol{F}$ is not invariant.

$\underline{ad\ \mathbf{w})}$: (I) $a = a^*$ iff $a = a^+$, and (II) $a^* \in \mathbf{BT}$ iff $a^+ \in \mathbf{BT}$.
We reason by induction on $a \in \mathbf{FT}$. If $a = \mathrm{t}$ or $a = \mathrm{e}$, $a^* = a = a^+$ and thus both (I) and (II) hold trivially. – If $a = (bc)$ and $a^* = (b^*c^*)$, then either: $b = b^*$, or: $c^* \in \mathbf{BT}$, and so either: $b = b^+$, or $b^+ \in \mathbf{BT}$, by induction. Hence $a^+ = (b^+c^+)$ and we have:

(I)	$a = a^*$	
iff	$b = b^*$ and $c = c^*$	characteristic property of ordered pairs
iff	$b = b^+$ and $c = c^+$	induction
iff	$a = a^+$	characteristic property of ordered pairs

(II)	$a^* \in \mathbf{BT}$	
iff	$c^* \in \mathbf{BT}$	def. **BT**
iff	$c^+ \in \mathbf{BT}$	induction
iff	$a^+ \in \mathbf{BT}$	def. **BT**

If $a = (bc)$ and $a^* \neq (b^*c^*)$, then $a^* = (b^*(c^*\mathrm{t}))$, $b \neq b^*$, and $c^* \notin \mathbf{BT}$, by the definition of $(bc)^*$. Hence, by induction: $b \neq b^+$, and $c^+ \notin \mathbf{BT}$, whence $a^+ = (b^+c^+(\mathrm{t}))$. So (I) holds because $a = (bc) \neq a^* = (b^*(c^*\mathrm{t}))$, given that $|c| \leq |c^*| < |c^*| + 1 = |(c^*\mathrm{t})$, and $a = (bc) \neq a^+ = (b^+(c^+\mathrm{t}))$, by the same token.[iv] And (II) holds given that both a^* and a^+ are Boolean. – If $a = (\mathrm{st})$, $a^* = a = a^+$ and thus both (I) and (II) hold trivially. – If $a = (\mathrm{se})$, $a^* = (\mathrm{e}(\mathrm{st})) = (\mathrm{e}^+(\mathrm{st})) = a^+$ and again both (I) and (II) hold trivially. – If $a = (\mathrm{s}(bc))$, where $b \neq \mathrm{s}$, then $a^* = (b^*(\mathrm{s}c^*)) \neq (\mathrm{s}(bc)) = a$; for otherwise we would have: $\mathrm{s} = b^* \in \mathbf{BT}$. But then $a^+ = ((bc)^+(\mathrm{st})) \neq (\mathrm{s}(bc)) = a$; for otherwise $\mathrm{s} = (bc)^+ \in \mathbf{RT}$. Hence a satisfies (I). Moreover, $a^* \in \mathbf{BT}$, by (i), and $a^+ = ((bc)^+(\mathrm{st})) \in \mathbf{BT}$, which settles (II).–Finally (and similarly), if $a = (\mathrm{s}(\mathrm{s}b))$, then $a^* = (\mathrm{s}b^*(\mathrm{s}b^*)(\mathrm{st}) \neq a$; for otherwise we would have: $\mathrm{s} = (\mathrm{s}b^) * \in \mathbf{BT}$. But then $a^+ = ((\mathrm{s}b)^+(\mathrm{st})) \neq (\mathrm{s}(\mathrm{s}b)) = a$; for otherwise $\mathrm{s} = (sb)^+ \in \mathbf{RT}$. Hence a satisfies (I). Moreover, $a^* \in \mathbf{BT}$, by (i), and $a^+ = ((\mathrm{s}b)^+(\mathrm{st})) \in \mathbf{BT}$, which settles (II) and completes the proof.

A.2 Footnotes

A.2.1 Fn. 14

We need to find a bijection $x \mapsto \tilde{x}$ between $\bigcup_{a \in \mathbf{FT}} Var_{a \smallsetminus \{i\}}$ and $\bigcup_{a \in \mathbf{RT}} Var_a$ such that, for any $a \in \mathbf{FT}$, $\tilde{x} \in Var_{a^*}$ if $x \in Var_a$. To construct such $\tilde{\cdot}$, we represent all variables of a type $a \in \mathbf{FT}$ as $v_{n,a}$, where $n \in \mathbb{N}$. First, we observe that, for any $c \in \mathbf{RT}$, the set $[c] := \{a \in \mathbf{FT} \mid a^* = c\}$ is finite, by **f)**, and non-empty, by **h)**, and can thus be represented by way of a (repetition-free) list of some length $\backslash c \backslash + 1$: $[c] = \{[c]_0, \ldots, [c]_{\backslash c \backslash}\} \neq \varnothing$. As a case in point, $[\mathrm{e}(\mathrm{st})] = \{\mathrm{e}(\mathrm{st}), \mathrm{s}(\mathrm{et}), \mathrm{se}\}$, whence $\backslash \mathrm{e}(\mathrm{st}) \backslash = 2$. Keeping the enumeration fixed

[iv] We leave it to the reader to show that, for any $a \in \mathbf{BT}, |a| \leq |a^+|$, which is just as straightforward as observation **f)**.

(for each $c \in \mathbf{RT}$), any type $a \in \mathbf{FT}$ is thus identifiable by its position $\sharp a$: $a = [a^*]_{\sharp a}$, where, $\sharp a \neq \sharp b$ if $a^* = b^*$ and $a \neq b$. Continuing the example, we thus have: $\sharp \mathrm{se} = 2$. We now put, for any $n \in \mathbb{N}$ and $a \in \mathbf{FT}$:

$\widetilde{v_{n,a}} := v_{f(n,a),a^*}$, where $f(n,a) = n \times (\backslash a^* \backslash + 1) + \sharp a$.

Using the same enumeration as before, we thus have: $\widetilde{v_{2,\mathrm{se}}} = v_{8,\mathrm{e(st)}}$; and the variables of type $a = \mathrm{e(st)}$ are arranged as follows, where $b = \mathrm{s(et)}$, and $c = \mathrm{se}$: $v_{0,a} = \widetilde{v_{0,a}}$; $v_{1,a} = \widetilde{v_{0,b}}$; $v_{2,a} = \widetilde{v_{0,c}}$; $v_{3,a} = \widetilde{v_{1,a}}$; $v_{4,a} = \widetilde{v_{1,b}}$; $v_{5,a} = \widetilde{v_{1,c}}$; etc.

Obviously, $\tilde{x} \in Var_{a^*}$ whenever $x \in Var_a$ and $a \in \mathbf{FT}$. It remains to be shown that $x \mapsto \tilde{x}$ is (a) one-one and (b) onto $\bigcup_{a \in \mathbf{RT}} Var_a$. <u>*ad* (a)</u>: if $x,\ y \in Var \smallsetminus \{i\}$ and $x = v_{n,a} \neq y = v_{m,b}$, where $a^* \neq b^*$, $\tilde{x} \in Var_{a^*}$ and $\tilde{y} \in Var_{b^*}$, and so $\tilde{x} \neq \tilde{y}$, since $Var_{a^*} \cap Var_{b^*} = \varnothing$.[v] But if $a^* = b^*$, and $x = v_{n,a} \neq v_{m,b} = y$, then either (a1) $\sharp a = \sharp b$, or (a2) $\sharp a \neq \sharp b$ but $m \neq n$. In either case we have $f(n,a) \neq f(m,b)$:

(a1): $f(n,a) = n \times (\backslash a^* \backslash + 1) + \sharp a \neq m \times (\backslash a^* \backslash + 1) + \sharp a = m \times (\backslash b^* \backslash + 1) + \sharp b = f(m,b)$;

(a2): $f(n,a)$ mod $a^* = \sharp a$, whereas $f(m,b)$ mod $a^* = f(m,b)$ mod $b^* = \sharp b$. and so $f(n,a) \neq f(m,b)$.[vi]

So in both cases we have: $\tilde{x} = v_{f(n,a),a^*} \neq v_{f(n,b),a^*} = v_{f(x,b),b^*} = \tilde{y}$.

<u>*ad* (b)</u>: If $c \in \mathbf{RT}$ and $y = v_{m,c}$, then either (b1) $m \leq \backslash c \backslash$, or (b2) $m > \backslash c \backslash$. If (b1), then there is exactly one $a \in [c]$ such that $m = \sharp a$ and thus $f(0,a) = 0 \times (\backslash a^* \backslash + 1) + \sharp a = m$, whence $\widetilde{v_{0,a}} = v_{f(0,a),a^*} = v_{m,c} = y$. And if (b2), then there is a $k \leq \backslash c \backslash$ such that $\backslash c \backslash$ mod $m = k$ and thus an $a \in [c]$ such that $\sharp a = k$; so if we put $n := \frac{m-k}{\backslash c \backslash + 1}$, we have: $f(n,a) = n \times (\backslash a^* \backslash + 1) + \sharp a = \frac{m-k}{\backslash c \backslash + 1} \times (\backslash c \backslash + 1) + k = m - k + k = m$, and thus: $\widetilde{v_{n,a}} = v_{f(n,a),a^*} = v_{m,c} = y$.

A.2.2 Fn. 30

<u>(i)</u>: (te)(ee) is logical.
This is because id_{e} is ee-invariant, and thus so is $\{(\boldsymbol{f}, id_{\mathrm{e}}) | \boldsymbol{f} \in D_{\mathrm{te}}\} [\in D_{\mathrm{(te)(ee)}}]$, as is readily verified.

<u>(ii)</u>: $D_{\mathrm{(te)(ee)}}$ contains an injective function.

In view of the basic set-theoretic fact that, for any $a, b \in \mathbf{2T}$, $\overline{\overline{D_{ab}}} = \overline{\overline{D_b}}^{\overline{\overline{D_a}}}$ (where $\overline{\overline{X}}$ is the cardinality of a set X), it must be shown that $\overline{\overline{D_{\mathrm{te}}}} \leq \overline{\overline{D_{\mathrm{ee}}}}$, for any $n := \overline{\overline{D_{\mathrm{e}}}}$. Now, if n is infinite, then $\overline{\overline{D_{\mathrm{te}}}} = n^2 = n < 2^n = n^n = \overline{\overline{D_{\mathrm{ee}}}}$, by basic cardinal arithmetic.[vii] And if n is finite, 3 cases may be distinguished:

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[v]This is one of the standard disjointness assumptions about logical notation that we have been making all along.

[vi]Here n mod k is the remainder of the (Euclidian) division of the dividend n by the divisor k.

[vii]See, e.g., Ebbinghaus [2, p. 142, 2.12].

$\underline{n = 1}$: Then $\overline{\overline{D_{\mathrm{te}}}} = \overline{\overline{D_{\mathrm{e}}}}^{\overline{\overline{D_{\mathrm{t}}}}} = 1^2 = 1 = 1^1 = \overline{\overline{D_{\mathrm{e}}}}^{\overline{\overline{D_{\mathrm{e}}}}}$.

$\underline{n = 2}$: Then $\overline{\overline{D_{\mathrm{e}}}} = \overline{\overline{D_{\mathrm{t}}}}$ and thus $\overline{\overline{D_{\mathrm{te}}}} = \overline{\overline{D_{\mathrm{ee}}}}$.

$\underline{n > 2}$: Then $n = m + 2$ (where $m > 0$) and:

$$\overline{\overline{D_{\mathrm{te}}}} = \overline{\overline{D_{\mathrm{e}}}}^{\overline{\overline{D_{\mathrm{t}}}}} = n^2 \leq n^m \times n^2 = n^{m+2} = n^n = \overline{\overline{D_{\mathrm{ee}}}}.$$

<u>(iii)</u>: (te) is not embeddable into (ee).
It suffices to find some D_e such that $D_{(\mathrm{te})(\mathrm{ee})}$ does not contain any invariant 1-1 function. So let $\overline{\overline{D_{\mathrm{e}}}}$ be 2. Then $\overline{\overline{D_{\mathrm{te}}}} = \overline{\overline{D_{\mathrm{ee}}}} = 2^2 = 4$ is finite and so any injective $\boldsymbol{F} \in D_{(\mathrm{te})(\mathrm{ee})}$ is surjective; in particular, $id_{\mathrm{e}} = \boldsymbol{F}(\boldsymbol{f})$, for some $\boldsymbol{f} \in D_{\mathrm{te}}$. But then, given such $\boldsymbol{F}$ and $\boldsymbol{f}$ and the permutation π_{e} that swaps the two members of D_{e}, we would also have:

	$\boldsymbol{F}(\boldsymbol{f})$	
=	$\pi_{\mathrm{ee}}(\boldsymbol{F}(\boldsymbol{f}))$	invariance of $\boldsymbol{F}(\boldsymbol{f})$ [$= id_{\mathrm{e}}$]
=	$\pi_{(\mathrm{te})(\mathrm{ee})}(\boldsymbol{F})(\pi_{\mathrm{te}}(\boldsymbol{f}))$	definition of $\pi_{(\mathrm{te})(\mathrm{ee})}$
=	$\boldsymbol{F}(\pi_{\mathrm{te}}(\boldsymbol{f}))$	invariance of $\boldsymbol{F}$

Given the injectivity of $\boldsymbol{F}$, it then follows that $\boldsymbol{f} = \pi_{\mathrm{te}}(\boldsymbol{f})$, which cannot be because:

	$\pi_{\mathrm{te}}(\boldsymbol{f})(1)$	
=	$\pi_{\mathrm{te}}(\boldsymbol{f})(\pi_{\mathrm{t}}(1))$	$\pi_{\mathrm{t}} = id_{\mathrm{t}}$
=	$\pi_{\mathrm{e}}(\boldsymbol{f}(1))$	definition of $\pi_{(\mathrm{te})(\mathrm{ee})}$
≠	$\boldsymbol{f}(1)$	definition of π_{e}

A.3 Lemmas

A.3.1 κ-Lemma

<u>Claim</u>: $\dot{\kappa}_a$ is injective (one-one).
We reason inductively on $a \in \mathbf{FT}$ and make use of the characterizations of $\dot{\kappa}_a$ observed in **k)**.

If $\boldsymbol{f}, \boldsymbol{f}' \in D_a$ and $\boldsymbol{f} \neq \boldsymbol{f}'$, where $a = bc$, then $\boldsymbol{f}(\boldsymbol{x}_0) \neq \boldsymbol{f}'(\boldsymbol{x}_0)$, for some $\boldsymbol{x}_0 \in D_b$. Hence (!) $\dot{\kappa}_c(\boldsymbol{f}(\boldsymbol{x}_0)) \neq \dot{\kappa}_c(\boldsymbol{f}'(\boldsymbol{x}_0))$, by induction. There are three cases:

- $b = b^*$:
  then, by (i), $\dot{\kappa}_c(\boldsymbol{f}(\boldsymbol{x}_0)) = \dot{\kappa}_a(\boldsymbol{f})(\boldsymbol{x}_0)$ and $\dot{\kappa}_c(\boldsymbol{f}'(\boldsymbol{x}_0)) = \dot{\kappa}_a(\boldsymbol{f}')(\boldsymbol{x}_0)$.
  So, given (!), $\dot{\kappa}_a(\boldsymbol{f})(\boldsymbol{x}_0) \neq \dot{\kappa}_a(\boldsymbol{f}')(\boldsymbol{x}_0)$ and thus $\dot{\kappa}_a(\boldsymbol{f}) \neq \dot{\kappa}_a(\boldsymbol{f}')$.

- $b \neq b^*$ and $\boldsymbol{c} \in \mathbf{BT}$:
  then, by (i), $\dot{\kappa}_c(\boldsymbol{f}(\boldsymbol{x}_0)) = \dot{\kappa}_a(\boldsymbol{f})(\dot{\kappa}_b(\boldsymbol{x}_0))$ and $\dot{\kappa}_c(\boldsymbol{f}'(\boldsymbol{x}_0)) = \dot{\kappa}_a(\boldsymbol{f}')(\dot{\kappa}_b(\boldsymbol{x}_0))$.
  So, given (!), $\dot{\kappa}_a(\boldsymbol{f})(\dot{\kappa}_b(\boldsymbol{x}_0)) \neq \dot{\kappa}_a(\boldsymbol{f}')(\dot{\kappa}_b(\boldsymbol{x}_0))$ and thus $\dot{\kappa}_a(\boldsymbol{f}) \neq \dot{\kappa}_a(\boldsymbol{f}')$.

- $b \neq b^*$ and $c \notin \mathbf{BT}$:
  then, given (!), $/\dot{\kappa}_c(\boldsymbol{f}(\boldsymbol{x}_0))/ \neq /\dot{\kappa}_c(\boldsymbol{f}'(\boldsymbol{x}_0))/$; but, by (i), $/\dot{\kappa}_c(\boldsymbol{f}(\boldsymbol{x}_0))/ = \dot{\kappa}_a(\boldsymbol{f})(\dot{\kappa}_b(\boldsymbol{x}_0))$ and $/\dot{\kappa}_c(\boldsymbol{f}'(\boldsymbol{x}_0))/ = \dot{\kappa}_a(\boldsymbol{f}')(\dot{\kappa}_b(\boldsymbol{x}_0))$. So, $\dot{\kappa}_a(\boldsymbol{f})(\dot{\kappa}_b(\boldsymbol{x}_0)) \neq \dot{\kappa}_a(\boldsymbol{f}')(\dot{\kappa}_b(\boldsymbol{x}_0))$ and thus $\dot{\kappa}_a(\boldsymbol{f}) \neq \dot{\kappa}_a(\boldsymbol{f}')$.

If $\boldsymbol{c}$, $\boldsymbol{c}' \in D_{\text{se}}$, and $\boldsymbol{c} \neq \boldsymbol{c}'$, then $\dot{\kappa}_{\text{se}}(\boldsymbol{c}) = \overleftarrow{\boldsymbol{c}} \neq \overleftarrow{\boldsymbol{c}'} = \dot{\kappa}_{\text{se}}(\boldsymbol{c}')$, as pointed out in fn. 6.

If $\boldsymbol{F}, \boldsymbol{F}' \in D_a$ and $\boldsymbol{F} \neq \boldsymbol{F}'$, where $a = \text{s}(bc)$ and $a^* = (a^*(\text{s}b^*))$, then $\boldsymbol{F}(\boldsymbol{w}_0)(\boldsymbol{x}_0) \neq \boldsymbol{F}'(\boldsymbol{w}_0)(\boldsymbol{x}_0)$, for some $\boldsymbol{w}_0 \in D_s$ and $\boldsymbol{x}_0 \in D_b$; thus:
$[\hat{\boldsymbol{w}} : \ \boldsymbol{F}(\boldsymbol{w})(\boldsymbol{x}_0)](\boldsymbol{w}_0) \neq [\hat{\boldsymbol{w}} : \ \boldsymbol{F}(\boldsymbol{w})(\boldsymbol{x}_0)](\boldsymbol{w}_0)$, and so:
$[\hat{\boldsymbol{w}} : \ \boldsymbol{F}(\boldsymbol{w})(\boldsymbol{x}_0)] \neq [\hat{\boldsymbol{w}} : \ \boldsymbol{F}'(\boldsymbol{w})(\boldsymbol{x}_0)]$.
By induction, we thus have:
(!!) $\dot{\kappa}([\hat{\boldsymbol{w}} : \ \boldsymbol{F}(\boldsymbol{w})(\boldsymbol{x}_0)]) \neq \dot{\kappa}([\hat{\boldsymbol{w}} : \ \boldsymbol{F}(\boldsymbol{w})(\boldsymbol{x}_0)])$.
Now, by (i), $\dot{\kappa}([\hat{\boldsymbol{w}} : \ \boldsymbol{F}(\boldsymbol{w})(\boldsymbol{x}_0)]) = \dot{\kappa}(\boldsymbol{F})(\dot{\kappa}(\boldsymbol{x}_0))$ and $\dot{\kappa}([\hat{\boldsymbol{w}} : \ \boldsymbol{F}'(\boldsymbol{w})(\boldsymbol{x}_0)]) = \dot{\kappa}(\boldsymbol{F}')(\dot{\kappa}(\boldsymbol{x}_0))$. So, given (!!), $\dot{\kappa}(\boldsymbol{F})(\dot{\kappa}(\boldsymbol{x}_0)) \neq \dot{\kappa}(\boldsymbol{F}')(\dot{\kappa}(\boldsymbol{x}_0))$ and thus $\dot{\kappa}(\boldsymbol{F}) \neq \dot{\kappa}(\boldsymbol{F}')$.

If $\boldsymbol{C}$, $\boldsymbol{C}' \in D_a$ and $\boldsymbol{C} \neq \boldsymbol{C}'$, where $a = \text{s}(\text{s}b)$, then $\boldsymbol{C}(\boldsymbol{w}_0) \neq \boldsymbol{C}'(\boldsymbol{w}_0)$, for some $\boldsymbol{w}_0 \in D_{\text{s}}$. Hence $\dot{\kappa}(\boldsymbol{C}(\boldsymbol{w}_0)) \neq \dot{\kappa}(\boldsymbol{C}'(\boldsymbol{w}_0))$, by induction, and so: $\boldsymbol{c} = \{(\boldsymbol{w},(\ \boldsymbol{C}(\boldsymbol{w}))) \mid \boldsymbol{w} \in D_{\text{s}}\} \neq \{(\boldsymbol{w}, \dot{\kappa}(\boldsymbol{C}'(\boldsymbol{w}))) | \boldsymbol{w} \in D_{\text{s}}\} = \boldsymbol{c}'$. But then, by the injectivity of $\overleftarrow{\cdot}$ (fn. 6): $\dot{\kappa}(\boldsymbol{C}) = \overleftarrow{\boldsymbol{c}} \neq \overleftarrow{\boldsymbol{c}'} = \dot{\kappa}(\boldsymbol{C}')$.

A.3.2 Γ-Lemma

To be shown: $\dot{\Gamma}_a$ = "$rge(\dot{\kappa}_a)$". The proof proceeds by induction on $a \in \mathbf{FT}$:[viii]
<u>"⊆"</u>: Given any $\boldsymbol{X} \in D_{a^*}$ such that $[\![\Gamma_a]\!](\boldsymbol{X}) = 1$ (where $a \in \mathbf{FT}$), we need to find some $\boldsymbol{x} \in D_{a^*}$ such that $\boldsymbol{X} = \dot{\kappa}_a(\boldsymbol{x})$.

– If $a \in \{\text{t, e, st}\}$, $\boldsymbol{x} = \boldsymbol{X}$.

– If $a = (bc)$, where $b = b^*$, and $[\![\Gamma_a]\!](\boldsymbol{F}) = 1$, then for any $\boldsymbol{x} \in D_b$, $[\![\Gamma_c]\!](\boldsymbol{F}(\boldsymbol{x})) = 1$, and thus by induction, $\boldsymbol{F}(\boldsymbol{x}) = \dot{\kappa}_c(\boldsymbol{y})$, for some $\boldsymbol{y} = \dot{\kappa}_c^{-1}(\boldsymbol{F}(\boldsymbol{x}))$, given that $\dot{\kappa}$ is injective. So we put:
$\boldsymbol{f} := \{(\boldsymbol{x}, \dot{\kappa}_c^{-1}(\boldsymbol{F}(\boldsymbol{x}))) \mid \boldsymbol{x} \in D_b\}$
and observe:
$\dot{\kappa}_a(\boldsymbol{f}) = \{(\boldsymbol{x}, \dot{\kappa}_c(\boldsymbol{f}(\boldsymbol{x}))) | \ \boldsymbol{x} \in D_b\} =$
$\{(\boldsymbol{x}, \dot{\kappa}_c(\dot{\kappa}_c^{-1}(\boldsymbol{F}(\boldsymbol{x})))) | \ \boldsymbol{x} \in D_b\} = \{(\boldsymbol{x}, \boldsymbol{F}(\boldsymbol{x})) | \ \boldsymbol{x} \in D_b\} = \boldsymbol{F}$.

– If $a = (bc)$, where $b \neq b^*$ but $c \in \mathbf{BT}$, and $[\![\Gamma_a]\!](\boldsymbol{F}) = 1$, then $[\![\Gamma_c]\!](\boldsymbol{F}(\boldsymbol{X})) = 1$, for any $\boldsymbol{X}$ such that $[\![\Gamma_b]\!](\boldsymbol{X}) = 1$; hence by induction, $\boldsymbol{F}(\boldsymbol{X}) \in rge(\dot{\kappa}_c)$ whenever $\boldsymbol{X} \in rge(\dot{\kappa}_b)$. We now put:
$\boldsymbol{f} := \{(\dot{\kappa}_b^{-1}(\boldsymbol{X}), \dot{\kappa}_c^{-1}(\boldsymbol{F}(\boldsymbol{X}))) \mid \boldsymbol{X} \in rge(\dot{\kappa}_b)\}$.
Clearly, $\boldsymbol{f} \in D_a$: any $\boldsymbol{x} \in D_b$ is of the form $\dot{\kappa}_b^{-1}(\boldsymbol{X})$, where $\boldsymbol{X} = \dot{\kappa}_b(\boldsymbol{x}) \in$

[viii] Though the presentation separates the directions, the induction is understood as treating them simultaneously, so that at each step the inductive hypothesis has access to the full biconditional.

$rge(\dot\kappa_b)$; and ${}_c^{-1}(\boldsymbol{F}(\boldsymbol{X}))) \in D_c$. It remains to be shown that $\dot\kappa_a(\boldsymbol{f}) = \boldsymbol{F}$, i.e., that for any $\boldsymbol{X} \in D_{b^*}$:

(!) $\quad \dot\kappa_a(\boldsymbol{f})(\boldsymbol{X}) = \boldsymbol{F}(\boldsymbol{X})$.

Two cases must be distinguished, according to whether $\boldsymbol{X} \in rge(\dot\kappa_b)$. If so, $\boldsymbol{X} = \dot\kappa_b(\boldsymbol{x})$, for some $\boldsymbol{x}\in D_b$, and we may conclude:

$${}_a(\boldsymbol{f})(\boldsymbol{X}) = {}_a(\boldsymbol{f})(\dot\kappa_b(\boldsymbol{x})) = {}_c(\boldsymbol{f}(\boldsymbol{x})) = {}_c(\boldsymbol{f}(\dot\kappa_b^{-1}(\boldsymbol{X}))) =$$
$$\dot\kappa_c(\dot\kappa_c^{-1}(\boldsymbol{F}(\boldsymbol{X}))) = \boldsymbol{F}(\boldsymbol{X}),$$

thus establishing the first case of (!). But if $\boldsymbol{X} \notin rge(\dot\kappa_b)$, then by induction, $[\![\Gamma_b]\!](\boldsymbol{X}) = 0$, and so,[ix] by the definitions of Γ_a and $\dot\kappa_a$: $\boldsymbol{F}(\boldsymbol{X}) = \mathbf{0}_{c^*} = \dot\kappa_a(\boldsymbol{f})(\boldsymbol{X})$.

– If $a = (bc), b \neq b^*, c \notin \mathbf{BT}$, and $[\![\Gamma_a]\!](\boldsymbol{F}) = 1$, then by induction, for any $\boldsymbol{X} \in rge(\dot\kappa_b)$, there is a $\boldsymbol{Y} \in rge(\dot\kappa_c)$ such that $\boldsymbol{F}(\boldsymbol{X}) = /\boldsymbol{Y}/$. We now put:

$$\boldsymbol{f} := \{(\dot\kappa_b^{-1}(\boldsymbol{X}), \boldsymbol{y}) \mid \boldsymbol{F}(\boldsymbol{X})(\dot\kappa_c(\boldsymbol{y})) = 1\}.$$

Clearly, $\boldsymbol{f} \in D_a$: any $\boldsymbol{x} \in D_b$ is of the form $\dot\kappa_b^{-1}(\boldsymbol{X})$, where $\boldsymbol{X} = \dot\kappa_b(\boldsymbol{x}) \in rge(\dot\kappa_b)$; and given that $\boldsymbol{F} \in D_a, \boldsymbol{y} \in D_c$ if $\boldsymbol{F}(\boldsymbol{X})(\dot\kappa_c(\boldsymbol{y})) = 1$.[x] Hence, for any $\boldsymbol{X} \in rge(\dot\kappa_b)$ and $\boldsymbol{Y} \in D_{c^*}$:

	$\dot\kappa_a(\boldsymbol{f})(\boldsymbol{X})(\boldsymbol{Y}) = 1$	
iff	$\boldsymbol{Y} = \dot\kappa_c(\boldsymbol{f}(\dot\kappa_b^{-1}(\boldsymbol{X})))$	by the definition of $\dot\kappa_c$
iff	$\dot\kappa_c^{-1}(\boldsymbol{Y}) = \boldsymbol{f}(\dot\kappa_b^{-1}(\boldsymbol{X}))$	by the injectivity of $\dot\kappa_c$
iff	$\boldsymbol{F}(\boldsymbol{X})(\dot\kappa_c(\dot\kappa_c^{-1}(\boldsymbol{Y}))) = 1$	by the definition of $\boldsymbol{f}$
iff	$\boldsymbol{F}(\boldsymbol{X})(\boldsymbol{Y}) = 1$	by the injectivity of $\dot\kappa_c$

On the other hand, if $\boldsymbol{F} \notin rge(\dot\kappa_b)$, then $[\![\Gamma_b]\!](\boldsymbol{X}) = 0$, by induction. Hence, as in the case before, the definitions of Γ_a and κ_a guarantee that $\boldsymbol{F}(\boldsymbol{X}) = \mathbf{0}_{c^*\mathrm{t}} = \dot\kappa_a(\boldsymbol{f})(\boldsymbol{X})$.

– If $a = (\mathrm{se})$ and $[\![\Gamma_a]\!](\boldsymbol{R}) = 1$, then for any $\boldsymbol{w} \in D_\mathrm{s}$ there is a unique $\boldsymbol{u} \in D_\mathrm{e}$ such that $\boldsymbol{R}(\boldsymbol{u})(\boldsymbol{w}) = 1$. So we put:

$$\boldsymbol{c} := \{(\boldsymbol{w}, \boldsymbol{u}) \in D_\mathrm{s} \times D_\mathrm{e} \mid \boldsymbol{R}(\boldsymbol{u})(\boldsymbol{w}) = 1\} \in D_{\mathrm{se}},$$

and we have, for any $\boldsymbol{u}_0 \in D_\mathrm{e}$ and $\boldsymbol{w}_0 \in D_\mathrm{s}$:

	${}_a(\boldsymbol{c})(\boldsymbol{u}_0)(\boldsymbol{w}_0) = 1$	
iff	$\overleftarrow{\boldsymbol{c}}(\boldsymbol{u}_0)(\boldsymbol{w}_0) = 1$	by the definition of $\dot\kappa_a$
iff	$\boldsymbol{c}(\boldsymbol{w}_0) = \boldsymbol{u}_0$	cf. fn. 6
iff	$\boldsymbol{R}(\boldsymbol{u}_0)(\boldsymbol{w}_0) = 1.$	by the definition of $\boldsymbol{c}$

Hence $\boldsymbol{R} = \dot\kappa_a(\boldsymbol{c})$, as required.

– If $a = (\mathrm{s}(bc)), and$ $[\![\Gamma_a]\!](\boldsymbol{H}) = 1$, then $[\![\Gamma_{\mathrm{sc}}]\!](\boldsymbol{H}(\boldsymbol{X})) = 1$, for any $\boldsymbol{X}$ such that $[\![\Gamma_b]\!](\boldsymbol{X}) = 1$. Hence by induction, $\boldsymbol{H}(\boldsymbol{X}) \in rge(\dot\kappa_{\mathrm{sc}})$ whenever $\boldsymbol{X} \in rge(\dot\kappa_b)$. We now put:

$$\boldsymbol{F} := [\hat{\boldsymbol{w}} : \{(\dot\kappa_b^{-1}(\boldsymbol{X}), \dot\kappa_{sc}^{-1}(\boldsymbol{H}(\boldsymbol{X}))(\boldsymbol{w})) \mid \boldsymbol{X} \in rge(\dot\kappa_b)\}] \qquad [\in D_a]$$

and it remains to be shown that:

(!!) $\quad \dot\kappa_a(\boldsymbol{F})(\boldsymbol{X}) = \boldsymbol{H}(\boldsymbol{X})$,

for any $\boldsymbol{X} \in D_{b^*}$. Now, two cases need to be distinguished:

[ix]Here the architectural remarks in fn. viii come to bear.

[x]The argument capitalizes on the simplification addressed in fn. 5.

Case 1: for some $\boldsymbol{x} \in D_b, \boldsymbol{X} = \dot{\kappa}_b(\boldsymbol{x})$; we then observe:

$$
\begin{array}{lll}
 & \dot{\kappa}_a(\boldsymbol{F})(\boldsymbol{X}) & \\
= & {}_a(\boldsymbol{F})(\dot{\kappa}_b(\boldsymbol{x})) & \boldsymbol{X} = \dot{\kappa}_b(\boldsymbol{x}) \\
= & {}_{\mathrm{sc}}([\hat{\boldsymbol{w}} : \boldsymbol{F}(\boldsymbol{w})(\boldsymbol{x})])) & \text{by the definition of } \dot{\kappa}_a \\
= & \dot{\kappa}_{\mathrm{sc}}([\hat{\boldsymbol{w}} : \dot{\kappa}_{sc}^{-1}(\boldsymbol{H}(\boldsymbol{X}))(\boldsymbol{w})]) & \boldsymbol{F}(\boldsymbol{w})(\boldsymbol{x}) \;=\; {}_{sc}^{-1}(\boldsymbol{H}(\boldsymbol{X}))(\boldsymbol{w}) \\
= & \dot{\kappa}_{\mathrm{sc}}(\dot{\kappa}_{sc}^{-1}(\boldsymbol{H}(\boldsymbol{X}))) & \text{by } \eta\text{-conversion}^{\mathrm{xi}} \\
= & \boldsymbol{H}(\boldsymbol{X}) & \dot{\kappa}_{sc} \circ \dot{\kappa}_{\mathrm{sc}}^{-1} = \boldsymbol{id}_{\mathrm{sc}}
\end{array}
$$

Case 2: $\boldsymbol{X} \notin rge(\dot{\kappa}_b)$, $\dot{\kappa}_a(\boldsymbol{F})(\boldsymbol{X}) = \mathbf{0}_{(\mathrm{sc})^*}$, by the definition of $\dot{\kappa}_a$, and $\boldsymbol{H}(\boldsymbol{X}) = \mathbf{0}_{(\mathrm{sc})^*}$, by the definition of Γ_a and the inductive hypothesis.

– If $a = (\mathrm{s}(\mathrm{s}b))$, where $b \in \mathbf{FT}$, and $[\![\Gamma_a]\!](\boldsymbol{R}) = 1$, then by induction, for any $\boldsymbol{w} \in D_{\mathrm{s}}$, there is a uniqe $\boldsymbol{X}_{\boldsymbol{w}} \in D_{(\mathrm{s}b)^*}$ such that $\boldsymbol{R}(\boldsymbol{X}_{\boldsymbol{w}})(\boldsymbol{w}) = 1$; moreover, $\boldsymbol{X}_{\boldsymbol{w}} \in rge(\dot{\kappa}_{\mathrm{s}b})$. We now put:

$$\boldsymbol{C} := [\hat{\boldsymbol{w}}_1 : [\hat{\boldsymbol{w}}_2 : \dot{\kappa}_{\mathrm{s}b}^{-1}(\boldsymbol{X}_{\boldsymbol{w}_1})(\boldsymbol{w}_2)]]$$

Clearly, $\boldsymbol{C} \in D_a$, and we need to show that, $\dot{\kappa}_a(\boldsymbol{C}) = \boldsymbol{R}$. And in fact, for any $\boldsymbol{Y} \in D_{(\mathrm{s}b)^*}$ and $\boldsymbol{w}_0 \in D_s$ we have:

$$
\begin{array}{lll}
 & \dot{\kappa}_a(\boldsymbol{C})(\boldsymbol{Y})(\boldsymbol{w}_0) = 1 & \\
\text{iff} & \overleftarrow{[\hat{\boldsymbol{w}} : \ \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{C}(\boldsymbol{w})]}(\boldsymbol{Y})(\boldsymbol{w}_0) = 1 & \text{by the definition of } \dot{\kappa}_a \\
\text{iff} & [\hat{\boldsymbol{w}} : \ \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{C}(\boldsymbol{w})](w_0) = \boldsymbol{Y} & \text{cf. fn 6} \\
\text{iff} & \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{C}(\boldsymbol{w}_0)) = \boldsymbol{Y} & \text{by } \beta\text{-conversion}^{\mathrm{xii}} \\
\text{iff} & \dot{\kappa}_{\mathrm{s}b}([\hat{\boldsymbol{w}}_1 : [\hat{\boldsymbol{w}}_2 : \dot{\kappa}_{(\mathrm{s}b)^*}^{-1}(\boldsymbol{X}_{\boldsymbol{w}_1})(\boldsymbol{w}_2)]](\boldsymbol{w}_0)) = \boldsymbol{Y} & \text{by the definition of } \boldsymbol{C} \\
\text{iff} & \dot{\kappa}_{\mathrm{s}b}([\hat{\boldsymbol{w}}_2 : \dot{\kappa}_{\mathrm{s}b}^{-1}(\boldsymbol{X}_{\boldsymbol{w}_0})(\boldsymbol{w}_2)]) = \boldsymbol{Y} & \text{by } \beta\text{-conversion} \\
\text{iff} & \dot{\kappa}_{\mathrm{s}b}(\dot{\kappa}_{\mathrm{s}b}^{-1}(\boldsymbol{X}_{\boldsymbol{w}_0})) = \boldsymbol{Y} & \text{by } \eta\text{-conversion} \\
\text{iff} & \boldsymbol{X}_{\boldsymbol{w}_0} = \boldsymbol{Y} & \dot{\kappa}_{\mathrm{s}b} \circ \dot{\kappa}_{\mathrm{s}b}^{-1} = \boldsymbol{id}_b \\
\text{iff} & \boldsymbol{R}(\boldsymbol{Y})(\boldsymbol{w}_0) = 1. & \text{by the definition of } \boldsymbol{X}_{\boldsymbol{w}_0}
\end{array}
$$

"$\supseteq$": We need to show that, for any $\boldsymbol{x} \in D_a, [\![\Gamma_a]\!](\dot{\kappa}_a(\boldsymbol{x})) = 1$. As in the forward direction, the base cases $a \in \{\mathrm{t}, \mathrm{e}, \mathrm{st}\}$ are trivial.

– If $a = (bc)$, where $b = b^*$, and $\boldsymbol{f} \in D_a$, we must show that $[\![\Gamma_a]\!](\dot{\kappa}_a(\boldsymbol{f})) = 1$, which, by the definition of Γ_a, means that for any $\boldsymbol{x} \in D_b, [\![\Gamma_c]\!](\dot{\kappa}_a(\boldsymbol{f})(\boldsymbol{x})) = 1$. By induction, this in turn means that ${}_a(\boldsymbol{f})(\boldsymbol{x}) \in rge(\dot{\kappa}_c)$. But then by the definition of $\dot{\kappa}_a : \dot{\kappa}_a(\boldsymbol{f})(\boldsymbol{x}) = \dot{\kappa}_c(\boldsymbol{f}(\boldsymbol{x}))$, and so $\dot{\kappa}_a(\boldsymbol{f})(\boldsymbol{x}) \in rge(\dot{\kappa}_c)$, indeed.

– If $a = (bc)$, where $b \neq b^*$ but $c \in \mathbf{BT}$, and $\boldsymbol{f} \in D_a$, it must be shown that, for any $\boldsymbol{X} \in D_{b^*}$: (1) $[\![\Gamma_c]\!](\dot{\kappa}_a(\boldsymbol{f})(\boldsymbol{X})) = 1$ if $[\![\Gamma_b]\!](\boldsymbol{X}) = 1$; and that (2) $\dot{\kappa}_a(\boldsymbol{f})(\boldsymbol{X})) = \mathbf{0}_{c^*}$ if $[\![\Gamma_b]\!](\boldsymbol{X}) = 0$. By induction, (1) means that $\dot{\kappa}_a(\boldsymbol{f})(\boldsymbol{X}) \in rge(\dot{\kappa}_c)$ if $\boldsymbol{X} = \dot{\kappa}_b(\boldsymbol{x})$, for some $\boldsymbol{x} \in D_b$; but then, by the definition of $\dot{\kappa}_a : \dot{\kappa}_a(\boldsymbol{f}(\dot{\kappa}_b(\boldsymbol{x})) = \dot{\kappa}_c(\boldsymbol{f}(\boldsymbol{X})) \in rge(\dot{\kappa}_c)$. And (2) holds because by induction, $[\![\Gamma_b]\!](\boldsymbol{X}) = 0$ implies: $\boldsymbol{X} \notin rge(\dot{\kappa}_b)$, and thus by the definition of $\dot{\kappa}_a : \dot{\kappa}_a(\boldsymbol{f})(\boldsymbol{X}) = \mathbf{0}_{c^*}$, indeed.

[xi] The functions $\boldsymbol{f}$ denoted by our meta-linguistic abstraction operator "$\hat{\boldsymbol{w}}$" are extensional and thus satisfy the equation: $[\hat{\boldsymbol{w}} : \boldsymbol{f}(\boldsymbol{w})] = \boldsymbol{f}$, aka η-conversion.

[xii] This is the logical principle, according to which $[\hat{\boldsymbol{w}} : \ldots \boldsymbol{w} \ldots](\boldsymbol{w}_0) = \ldots \boldsymbol{w}_0 \ldots$, provided that $\boldsymbol{w}_0$ does not get bound in the substitution process.

– If $a = (bc), b \neq b^*, c \notin \mathbf{BT}$, and $\boldsymbol{f} \in D_a$, it must be shown that (1) for any $\boldsymbol{X} \in rge(\dot{\kappa}_b) : \dot{\kappa}_a(\boldsymbol{f})(\boldsymbol{X}) = /\boldsymbol{Y}/$, for some $\boldsymbol{Y} \in rge(\dot{\kappa}_c)$; and that (2) $\dot{\kappa}_a(\boldsymbol{f})(\boldsymbol{X}) = \mathbf{0}_{c^*\mathrm{t}}$ whenever $\boldsymbol{X} \in D_{b^*} \setminus rge(\dot{\kappa}_b)$. (2) follows as in the previous case. In case (1), $\boldsymbol{X} = \dot{\kappa}_b(\boldsymbol{X})$ again, whence, $\dot{\kappa}_a$, we have, following the definition of $\dot{\kappa}_a : \dot{\kappa}_a(\boldsymbol{f})(X) = /\dot{\kappa}_c(\boldsymbol{f}(\boldsymbol{x}))/$, which is of the required form.

– If $a = (\mathrm{se})$ and $\boldsymbol{c} \in D_a$, we must show that, for any $\boldsymbol{w} \in D_w$, there is exactly one $\boldsymbol{u_w} \in D_e$ such that: $\dot{\kappa}_a(\boldsymbol{c})(\boldsymbol{u_w})(\boldsymbol{w}) = 1$. If we put: $\boldsymbol{u_w} = \boldsymbol{c}(\boldsymbol{w})$, the definition of $\dot{\kappa}_a$ yields. $\dot{\kappa}_a(\boldsymbol{c})(\boldsymbol{u_w})(\boldsymbol{w}) = 1$ iff $\boldsymbol{c}(\boldsymbol{w}) = \boldsymbol{u_w}$, which is how we just defined $\boldsymbol{u_w}$; and its uniqueness follows because, for any $\boldsymbol{v} \neq \boldsymbol{u_w}$, we get: $\dot{\kappa}_a(\boldsymbol{c})(\boldsymbol{v})(\boldsymbol{w}) = 1$ iff $\boldsymbol{c}(\boldsymbol{w}) = \boldsymbol{v}$, which is not the case, by assumption. – If $a = (\mathrm{s}(bc))$ and $\boldsymbol{F} \in D_a$, it must be shown that (1) for any $\boldsymbol{X} \in rge(\dot{\kappa}_b) : \dot{\kappa}_a(\boldsymbol{F})(\boldsymbol{X}) \in rge(\dot{\kappa}_{\mathrm{s}b})$; and that (2) $\dot{\kappa}_a(\boldsymbol{F})(\boldsymbol{X}) = \mathbf{0}_{(\mathrm{s}c)^*}$ whenever $\boldsymbol{X} \in D_{b^*} \setminus rge(\dot{\kappa}_b)$. Again, (1) holds because $\boldsymbol{X} = \dot{\kappa}_b(\boldsymbol{x})$, where $\boldsymbol{x} \in D_b$, and thus $\dot{\kappa}_a(\boldsymbol{F})(\boldsymbol{X}) = \dot{\kappa}_{\mathrm{s}c}([\hat{\boldsymbol{w}} : \ \boldsymbol{F}(\boldsymbol{w})(\boldsymbol{x})])$. And (2) follows as in previous cases.

– If $a = (\mathrm{s}(\mathrm{s}b))$ and $\boldsymbol{C} \in D_a$, we need to show that, for any $\boldsymbol{w} \in D_\mathrm{s}$, there is a unique $\boldsymbol{X_w} \in D_{(\mathrm{s}b)^*}$ such that $\dot{\kappa}_a(\boldsymbol{C})(\boldsymbol{X_w})(\boldsymbol{w}) = 1$ where $\boldsymbol{X_w} \in rge(\dot{\kappa}_{\mathrm{s}b})$. We put: $\boldsymbol{X_w} := \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{C}(\boldsymbol{w}))$ and observe that $\boldsymbol{X_w} \in rge(\dot{\kappa}_{\mathrm{s}b})$ and that:

$\overleftarrow{\dot{\kappa}_a(\boldsymbol{C})(\boldsymbol{X_w})(\boldsymbol{w})} = 1$

iff $[\hat{\boldsymbol{w}'} : \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{C}(\boldsymbol{w}'))](\boldsymbol{X_w})(\boldsymbol{w}) = 1$ by the definition of $\dot{\kappa}_a$ (and α-conversion)[xiii]

iff $[\hat{\boldsymbol{w}'} : \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{C}(\boldsymbol{w}'))](\boldsymbol{w}) = \boldsymbol{X_w}$ cf. fn. 6

iff $\dot{\kappa}_{\mathrm{s}b}(\boldsymbol{C}(\boldsymbol{w})) = \boldsymbol{X_w}$ by β-conversion,

which is how $\boldsymbol{X_w}$ was defined. And again, its uniqueness follows as a corollary of the above calculation, trading $\boldsymbol{X_w}$ for arbitrary $\boldsymbol{Y} \in D_{(\mathrm{s}b)^*}$.

A.3.3 Φ-Lemma

To be shown:

$$\begin{aligned}&\dot{\Phi}_{b,c}(\boldsymbol{F})(\boldsymbol{X})(\boldsymbol{Y}) = 1 \text{ iff}\\ &\text{there are } \boldsymbol{f} \in D_{bc}, \boldsymbol{x} \in D_b, \text{ and } \boldsymbol{y} \in D_c \text{ such that:}\\ &\qquad\qquad \boldsymbol{F} = \dot{\kappa}_{ab}(\boldsymbol{f}), \boldsymbol{X} = \dot{\kappa}_b(\boldsymbol{x}), \boldsymbol{Y} = \dot{\kappa}_c(\boldsymbol{y}), \text{ and } \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}.\end{aligned}$$

If $\dot{\Phi}_{b,c}(\boldsymbol{F})(\boldsymbol{X})(\boldsymbol{Y}) = 1$, two cases are to be distinguished. First, if $(bc)^* = b^*c^*$, then $\dot{\Gamma}_{bc}(\boldsymbol{F}) = \dot{\Gamma}_b(\boldsymbol{X}) = 1$ and $\boldsymbol{F}(\boldsymbol{X}) = \boldsymbol{Y}$, by the interpretation of IL^*. Hence $\boldsymbol{F} \in rge(\dot{\kappa}_{bc})$ and $\boldsymbol{X} \in rge(\dot{\kappa}_b)$, by the Γ -Lemma. So there are $\boldsymbol{f} \in D_{bc}$ and $\boldsymbol{x} \in D_b$ such that $\boldsymbol{F} = \dot{\kappa}_{bc}(\boldsymbol{f})$ and $\boldsymbol{X} = \dot{\kappa}_b(\boldsymbol{x})$, as required. Moreover, $\boldsymbol{Y} = \boldsymbol{F}(\boldsymbol{X}) = \dot{\kappa}_{bc}(\boldsymbol{f})(\dot{\kappa}_b(\boldsymbol{x})) = \dot{\kappa}_c(\boldsymbol{f}(\boldsymbol{x}))$, by the definition of $\dot{\kappa}_{bc}$, and thus the lemma holds for $\boldsymbol{y} := \boldsymbol{f}(\boldsymbol{x})$. The second case is similar: $\boldsymbol{f}$ and $\boldsymbol{x}$ exist by the Γ -Lemma, and if we put $\boldsymbol{y} := \boldsymbol{f}(\boldsymbol{x})$ again, the definition of $\dot{\kappa}_{bc}$ gives us: $\boldsymbol{F}(\boldsymbol{X}) = \dot{\kappa}_{bc}(\boldsymbol{f})(\dot{\kappa}_b(\boldsymbol{x})) = /\dot{\kappa}_c(\boldsymbol{f}(\boldsymbol{x}))/ = /\dot{\kappa}_c(\boldsymbol{y})/ = /\boldsymbol{Y}/$, by the third conjunct of Φ_{bc} – and thus: $\dot{\kappa}_c(\boldsymbol{y}) = \boldsymbol{Y}$, since $/\cdot/$ is (obviously) injective.

[xiii] In other words, the variable bound by the functional abstraction "$\hat{\boldsymbol{w}}$" may be renamed, if this causes no confusion.

A.3.4 Ψ-Lemma

To be shown: $\dot{\Psi}_a(\boldsymbol{C})(\boldsymbol{X})(\boldsymbol{w}) = 1$ iff

there are $\boldsymbol{c} \in D_{\mathrm{s}a}$ and $\boldsymbol{x} \in D_a$ such that:

(A) $\boldsymbol{C} = \dot{\kappa}_{\mathrm{s}a}(\boldsymbol{c})$, (B) $\boldsymbol{X} = \dot{\kappa}_a(\boldsymbol{x})$, and (C) $\boldsymbol{c}(\boldsymbol{w}) = \boldsymbol{x}$.

We proceed by induction on $|b|$.

– If $a = \mathrm{t}$, $\boldsymbol{p} \in D_{\mathrm{st}}, \boldsymbol{v} \in D_{\mathrm{t}}$ and $\boldsymbol{w}_0 \in D_{\mathrm{s}}$, then, by the interpretation of IL^* : $\dot{\Psi}_b(\boldsymbol{p})(\boldsymbol{v})(\boldsymbol{w}_0) = 1$ just in case (C) $\boldsymbol{p}(\boldsymbol{w}_0) = \boldsymbol{v}$, where (A) $\dot{\kappa}_{\mathrm{st}}(\boldsymbol{p}) = \boldsymbol{p}$ and (B) $\dot{\kappa}_{\mathrm{t}}(\boldsymbol{v}) = \boldsymbol{v}$, due to the characterization **k)** of $\dot{\kappa}_{\mathrm{t}}$ and $\dot{\kappa}_{\mathrm{st}}$.

– If $a = \mathrm{e}, \boldsymbol{R} \in D_{\mathrm{e(st)}}, \boldsymbol{u} \in D_{\mathrm{e}}$ and $\boldsymbol{w}_0 \in D_{\mathrm{s}}$, then, by the interpretation of IL^* : $\dot{\Psi}_b(\boldsymbol{R})(\boldsymbol{u})(\boldsymbol{w}_0) = 1$ just in case $\dot{\Gamma}(\boldsymbol{R})(\boldsymbol{w}_0) = \boldsymbol{R}(\boldsymbol{u})(\boldsymbol{w}_0) = 1$, which by the Γ-Lemma, means that, for some $\boldsymbol{c} \in D_{\mathrm{se}}$: $\boldsymbol{R} = \dot{\kappa}_{\mathrm{se}}(\boldsymbol{c})$ and $\boldsymbol{R}(\boldsymbol{u})(\boldsymbol{w}_0) = 1$. In other words: $\dot{\Psi}_b(\boldsymbol{R})(\boldsymbol{u})(\boldsymbol{w}_0) = 1$ iff there is a $\boldsymbol{c} \in D_{\mathrm{se}}$ such that: (A) $\boldsymbol{R} = \dot{\kappa}_{\mathrm{se}}(\boldsymbol{c})$, (B) $\boldsymbol{u} = \dot{\kappa}_{\mathrm{e}}(\boldsymbol{u})$, and $\dot{\kappa}_{(\mathrm{se})}(\boldsymbol{c})(\dot{\kappa}_{\mathrm{e}}(\boldsymbol{u}))(\boldsymbol{w}_0) = 1$, i.e.: (C) $\boldsymbol{c}(\boldsymbol{w}_0) = \boldsymbol{u}$, by the characterization **l)** of $\dot{\kappa}_{\mathrm{se}}$.

– If $a = bc, \boldsymbol{G} \in D_{(\mathrm{s}(bc))^*}, \boldsymbol{F} \in D_{(bc)^*} = D_{b^*c^*}$ and $\boldsymbol{w}_0 \in D_{\mathrm{s}}$, then $(\mathrm{s}(bc))^* = b^*(\mathrm{s}c)^*$, and:

$\dot{\Psi}_{bc}(\boldsymbol{G})(\boldsymbol{F})(\boldsymbol{w}_0) = 1$

iff $\dot{\Gamma}_{\mathrm{s}(bc)}(\boldsymbol{G}) = \dot{\Gamma}_{bc}(\boldsymbol{F}) = \dot{\Psi}_c(\boldsymbol{G}(\boldsymbol{X}))(\boldsymbol{F}(\boldsymbol{X}))(\boldsymbol{w}_0) = 1$
whenever $\dot{\Gamma}_b(\boldsymbol{X}) = 1$ — interpretation of IL^*

iff $\boldsymbol{G} \in rge(\dot{\kappa}_{\mathrm{s}(bc)}), \boldsymbol{F} \in rge(\dot{\kappa}_{bc})$, and $\dot{\Psi}_c(\boldsymbol{G}(\boldsymbol{X}))(\boldsymbol{F}(\boldsymbol{X}))(\boldsymbol{w}_0) = 1$
whenever $\boldsymbol{X} \in rge(\dot{\kappa}_b)$ — Γ-Lemma

iff there are $\boldsymbol{g} \in D_{\mathrm{s}(bc)}$ and $\boldsymbol{f} \in D_{bc}$ such that: $\dot{\kappa}_{\mathrm{s}(bc)}(\boldsymbol{g}) = \boldsymbol{G}$, $\dot{\kappa}_{bc}(\boldsymbol{f}) = \boldsymbol{F}$,
and for any $\boldsymbol{x} \in D_b$ there are $\boldsymbol{z} \in D_{\mathrm{sc}}$ and $\boldsymbol{y} \in D_c$ such that:
$\dot{\kappa}_{\mathrm{sc}}(\boldsymbol{z}) = \boldsymbol{G}(\dot{\kappa}_b(\boldsymbol{x}))$, $\dot{\kappa}_c(\boldsymbol{y}) = \boldsymbol{F}(\dot{\kappa}_b(\boldsymbol{x}))$, and $\boldsymbol{y} = \boldsymbol{z}(\boldsymbol{w}_0)$ — induction

iff there are $\boldsymbol{g} \in D_{\mathrm{s}(bc)}$ and $\boldsymbol{f} \in D_{bc}$ such that: $\dot{\kappa}_{\mathrm{s}(bc)}(\boldsymbol{g}) = \boldsymbol{G}$, $\dot{\kappa}_{bc}(\boldsymbol{f}) = \boldsymbol{F}$,
and for any $\boldsymbol{x} \in D_b$ there is a $\boldsymbol{z} \in D_{\mathrm{sc}}$ such that:
$\dot{\kappa}_{\mathrm{sc}}(\boldsymbol{z}) = \dot{\kappa}_{\mathrm{s}(bc)}(\boldsymbol{g})(\dot{\kappa}_b(\boldsymbol{x}))$ and $\dot{\kappa}_c(\boldsymbol{z}(\boldsymbol{w}_0)) = \dot{\kappa}_{bc}(\boldsymbol{f})(\dot{\kappa}_b(\boldsymbol{x}))$ — identity

iff there are $\boldsymbol{g} \in D_{\mathrm{s}(bc)}$ and $\boldsymbol{f} \in D_{bc}$ such that: $\dot{\kappa}_{\mathrm{s}(bc)}(\boldsymbol{g}) = \boldsymbol{G}$, $\dot{\kappa}_{bc}(\boldsymbol{f}) = \boldsymbol{F}$,
and for any $\boldsymbol{x} \in D_b$ there is a $\boldsymbol{z} \in D_{\mathrm{sc}}$ such that:
$\dot{\kappa}_{\mathrm{sc}}(\boldsymbol{z}) = \dot{\kappa}_{\mathrm{sc}}([\hat{\boldsymbol{w}} : \boldsymbol{g}(\boldsymbol{w})(\boldsymbol{x})])$ and $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{z}(\boldsymbol{w}_0)$ iff — def. $\dot{\kappa}_{\mathrm{s}(bc)}, \dot{\kappa}_{bc}$

iff there are $\boldsymbol{g} \in D_{\mathrm{s}(bc)}$ and $\boldsymbol{f} \in D_{bc}$ such that: $\dot{\kappa}_{\mathrm{s}(bc)}(\boldsymbol{g}) = \boldsymbol{G}$, $\dot{\kappa}_{bc}(\boldsymbol{f}) = \boldsymbol{F}$,
and for any $\boldsymbol{x} \in D_b$ there is a $\boldsymbol{z} \in D_{\mathrm{sc}}$ such that:
$\boldsymbol{z} = [\hat{\boldsymbol{w}} : \boldsymbol{g}(\boldsymbol{w})(\boldsymbol{x})]$ and $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{z}(\boldsymbol{w}_0)$ — injectivity of $\dot{\kappa}_{\mathrm{sc}}$

iff there are $\boldsymbol{g} \in D_{\mathrm{s}(bc)}$ and $\boldsymbol{f} \in D_{bc}$ such that: $\dot{\kappa}_{\mathrm{s}(bc)}(\boldsymbol{g}) = \boldsymbol{G}$, $\dot{\kappa}_{bc}(\boldsymbol{f}) = \boldsymbol{F}$,
and for any $\boldsymbol{x} \in D_b : \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{w}_0)(\boldsymbol{x})$ — identity + β-conversion

iff there are $\boldsymbol{g} \in D_{\mathrm{s}(bc)}$ and $\boldsymbol{f} \in D_{bc}$ such that:
(A) $\dot{\kappa}_{\mathrm{s}(bc)}(\boldsymbol{g}) = \boldsymbol{G}$, (B) $\dot{\kappa}_{bc}(\boldsymbol{f}) = \boldsymbol{F}$, and (C) $\boldsymbol{f} = \boldsymbol{g}(\boldsymbol{w}_0)$ — extensionality

– If $a = \mathrm{s}b$, $\boldsymbol{G} \in D_{(\mathrm{s}b)^*(\mathrm{st})}$, $\boldsymbol{C} \in D_{(\mathrm{s}b)^*}$ and $\boldsymbol{w}_0 \in D_{\mathrm{s}}$, then:

$\dot{\Psi}_{\mathrm{s}b}(\boldsymbol{G})(\boldsymbol{C})(\boldsymbol{w}_0) = 1$

iff $\dot{\Gamma}_{\mathrm{s}(\mathrm{s}b)}(\boldsymbol{G}) = \dot{\Gamma}_{\mathrm{s}b}(\boldsymbol{C}) = \boldsymbol{G}(\boldsymbol{C})(\boldsymbol{w}_0) = 1$ — interpretation of IL^*

iff there are $\boldsymbol{g} \in D_{\mathrm{s}(\mathrm{s}b)}$ and $\boldsymbol{c} \in D_{\mathrm{s}b}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}(\mathrm{s}b)}(\boldsymbol{g})$, $\boldsymbol{C} = \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{c})$ and $\boldsymbol{G}(\boldsymbol{C})(\boldsymbol{w}_0) = 1$ — Γ-Lemma

iff there are $\boldsymbol{g} \in D_{\mathrm{s}(\mathrm{s}b)}$ and $\boldsymbol{c} \in D_{\mathrm{s}b}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}(\mathrm{s}b)}(\boldsymbol{g})$, $\boldsymbol{C} = \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{c})$ and $\dot{\kappa}_{\mathrm{s}(\mathrm{s}b)}(\boldsymbol{g})(\dot{\kappa}_{\mathrm{s}b})(\boldsymbol{c}))(\boldsymbol{w}_0) = 1$ — identity

iff there are $\boldsymbol{g} \in D_{\mathrm{s}(\mathrm{s}b)}$ and $\boldsymbol{c} \in D_{\mathrm{s}b}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}(\mathrm{s}b)}(\boldsymbol{g})$, $\boldsymbol{C} = \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{c})$ and $\overleftarrow{[\hat{\boldsymbol{w}} : \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{g}(\boldsymbol{w}))]}(\dot{\kappa}_{\mathrm{s}b}(\boldsymbol{c}))(\boldsymbol{w}_0) = 1$ — def. $\dot{\kappa}_{\mathrm{s}(\mathrm{s}b)}$

iff there are $\boldsymbol{g} \in D_{\mathrm{s}(\mathrm{s}b)}$ and $\boldsymbol{c} \in D_{\mathrm{s}b}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}(\mathrm{s}b)}(\boldsymbol{g})$, $\boldsymbol{C} = \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{c})$ and $[\hat{\boldsymbol{w}} : \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{g}(\boldsymbol{w}))](\boldsymbol{w}_0) = \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{c})$ — e)

iff there are $\boldsymbol{g} \in D_{\mathrm{s}(\mathrm{s}b)}$ and $\boldsymbol{c} \in D_{\mathrm{s}b}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}(\mathrm{s}b)}(\boldsymbol{g})$, $\boldsymbol{C} = \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{c})$ and $\dot{\kappa}_{\mathrm{s}b}(\boldsymbol{g}(\boldsymbol{w}_0)) = \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{c})$ — β-conversion

iff there are $\boldsymbol{g} \in D_{\mathrm{s}(\mathrm{s}b)}$ and $\boldsymbol{c} \in D_{\mathrm{s}b}$ such that:
(A) $\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}(\mathrm{s}b)}(\boldsymbol{g})$, (B) $\boldsymbol{C} = \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{c})$ and (C) $\boldsymbol{g}(\boldsymbol{w}_0) = \boldsymbol{c}$ — injectivity of $\dot{\kappa}_{\mathrm{s}b}$

A.4 ω-Lemma

To be shown:

(Ω) $\omega(a^*) \le \omega(a^+)$.

If a =t or a =e, $a^* = a = a^+$ and thus (Ω) holds trivially. – If $a = (bc)$ (where $b \neq$s), then either (A) $a^+ = (b^+c^+)$ or else: (B) $a^+ = (b^+(c^+\mathrm{t}))$. In case (A), $b = b^+$ or $c^+ \in \mathbf{BT}$, and thus: $b = b^*$ or $c^* \in \mathbf{BT}$, by **z)**. So $a^* = (b^*c^*)$ and we have: $\omega(a^+) = max(\omega(b^+)+1, \omega(c^+))$ and $\omega(a^*) = max(\omega(b^*)+1, \omega(c^*))$. But then $\omega(a^*) \le \omega(a^+)$ because, by induction: $\omega(c^*) \le \omega(c^+)$ and $\omega(b^*) \le \omega(b^+)$ and so $\omega(b^*) + 1 \le \omega(b^+) + 1$. In Case (B), $a^* = (b^*(c^*\mathrm{t}))$, again by **z)**, and so we have:

$$\begin{aligned}
&\omega(a^+)\\
=\;& max(\omega(b^+) + 1, max(\omega(c^+) + 1, \omega(\mathrm{t})))\\
=\;& max(\omega(b^+) + 1, max(\omega(c^+) + 1, 0))\\
=\;& max(\omega(b^+) + 1, \omega(c^+) + 1);
\end{aligned}$$

and similarly:

$$\omega(a^*) = max(\omega(b^*) + 1, \omega(c^*) + 1).$$

But then, again, $\omega(a^*) \le \omega(a^+)$ because, by induction: $\omega(b^*) \le \omega(b^+)$ and $\omega(c^*) \le \omega(c^+)$ and so $\omega(b^*)+1 \le \omega(b^+)+1$ and $\omega(c^*)+1 \le \omega(c^+)+1$. – If $a =$ (st) or $a =$ (se), $a^* = a = a^+$ and thus thus (Ω) holds trivially. – If $a = \mathrm{s}(bc)$ (where $b \neq \mathrm{s}$), then:

$$\begin{aligned}
&\omega(a^*)\\
=\;& \omega(b^*(\mathrm{s}c)^*)\\
=\;& max(\omega(b^*) + 1, \omega((\mathrm{s}c)^*))\\
\le\;& max(\omega(b^+) + 1, \omega((\mathrm{s}c)^+)) && \text{by induction}\\
=\;& max(\omega(b^+) + 1, max(\omega((c)^+) + 1, \omega(\mathrm{st}))) && \text{def. } \omega((\mathrm{s}c)^+))\\
=\;& max(\omega(b^+) + 1, \omega((c)^+) + 1)
\end{aligned}$$

Moreover:

$$\begin{array}{ll} & \omega(a^+) \\ = & \omega((bc)^+(\mathrm{st})) \\ = & max(\omega((bc)^+) + 1, \max(\omega(\mathrm{s}) + 1), \omega(\mathrm{t}))) \\ = & max(\omega((bc)^+) + 1, 1) \\ = & \omega((bc)^+) + 1 \end{array}$$

It thus remains to be shown that $(+)$ $max(\omega(b^+)+1, \omega((c)^+)+1) \leq \omega((bc)^+)+1$, i.e., that $max(\omega(b^+), \omega(c^+)) \leq \omega((bc)^+)$. There are two cases:

- If $(bc)^+ = (b^+c^+)$, then $\omega((bc)^+) = max(\omega(b^+) + 1, \omega(c^+))$, and so we have both: $\omega(b^+) \leq \omega(b^+) + 1 \leq max(\omega(b^+) + 1, \omega(c^+)) = \omega((bc)^+)$ and: $\omega(c^+) \leq max(\omega(b^+) + 1, \omega(c^+)) = \omega((bc)^+)$, from which $(+)$ follows.

- If $(bc)^+ = (b^+(c^+\mathrm{t}))$, then $\omega((bc)^+) = max(\omega(b^+) + 1, \omega(c^+) + 1)$, and again we have both: $\omega(b^+) \leq \omega(b^+) + 1 \leq max(\omega(b^+) + 1, \omega(c^+) + 1) = \omega(c^+)$ and: $\omega(c^+) \leq \omega(c^+) + 1 \leq max(\omega(b^+) + 1, \omega(c^+) + 1) = \omega(c^+)$, which again implies $(+)$. –

Finally, if $a = \mathrm{s}(\mathrm{s}b)$, then:

$$\begin{array}{lll} & \omega(a^*) & \\ = & \omega((\mathrm{s}b)^*(\mathrm{st})) & \\ = & max(\omega(\mathrm{s}b)^* + 1, \omega(\mathrm{st})) & \\ = & max(\omega(\mathrm{s}b)^* + 1, 1) & \\ = & \omega(\mathrm{s}b)^* + 1 & \\ \leq & \omega(\mathrm{s}b)^+ + 1 & \text{by induction} \\ = & max(\omega(\mathrm{s}b)^+ + 1, 1) & \\ = & max(\omega(\mathrm{s}b)^+ + 1, \omega(\mathrm{st})) & \\ = & \omega((\mathrm{s}b)^+(\mathrm{st})) & \\ = & \omega((\mathrm{s}(\mathrm{s}b))^+) & \\ = & \omega(a^+) & \end{array}$$

...which completes the proof of (Ω).

A.5 $^{\mathrm{o}}$-Lemma

To be shown: $(\mathrm{s}a)$ is embeddable into $(\mathrm{s}a)^{\mathrm{o}}$, for any $a \in \mathbf{FT}$.
We first define a family of closed terms $\varepsilon_a \in \mathbf{IL}^*_{(\mathrm{s}a)(\mathrm{s}a^{\mathrm{o}})}$ (where $a \in \mathbf{FT}$) and then show that each $\dot{\varepsilon}_a$ is injective, which implies embeddability, given observation **s)**. The following recursion defines the ε-terms:

(a) If $a = \mathrm{t}$ and hence $(\mathrm{s}a)^{\mathrm{o}} = \mathrm{st}$, $\varepsilon_a := (\lambda p^{\mathrm{st}}.\ p)$ $[= \kappa_{\mathrm{st}}]$;

(b) if $a = \mathrm{e}$ and hence $(\mathrm{s}a)^{\mathrm{o}} = \mathrm{e}(\mathrm{st})$,
$$\varepsilon_a := (\lambda f^{\mathrm{se}}.\ (\lambda x^{\mathrm{e}}.\ (\lambda i.\ (f(i) = x)))) \ [= \kappa_{\mathrm{se}}];$$

(c) if $a = (\mathrm{s}b)$ and hence $(\mathrm{s}a)^{\mathrm{o}} = (\mathrm{s}b)^{\mathrm{o}}(\mathrm{st})$, two cases must be distinguished:

(c_1) if $b = \mathrm{t}$, then: $(\mathrm{s}b)^{\mathrm{o}} = (\mathrm{st})$ and:
$$\varepsilon_a := (\lambda C^{\mathrm{s(st)}}.\ (\lambda p^{\mathrm{st}}.\ (\lambda i.\ (C(i) = p))));$$

(c_2) if $b \neq \mathrm{t}$, then: $(\mathrm{s}b)^{\mathrm{o}} = (b^{\mathrm{o}}(\mathrm{st}))$ and:
$$\varepsilon_a := (\lambda C^{\mathrm{s}(\mathrm{s}b)}.\ (\lambda y^{(\mathrm{s}b)^{\mathrm{o}}}.\ (\lambda i.\ (\varepsilon_b(C(i)) = y))));$$

(d) if $a = (bc)$, where $b \neq \mathrm{s}$, and hence $(\mathrm{s}a)^{\mathrm{o}} = (bc)^{\mathrm{o}}(\mathrm{st}) = (b^{\mathrm{o}}c^{\mathrm{o}})(\mathrm{st})$, four cases must be distinguished:

(d_1) if $b = \mathrm{t} = c$, then: $(\mathrm{s}b)^{\mathrm{o}} = (\mathrm{s}c)^{\mathrm{o}} = \mathrm{st}$, $(\mathrm{s}(bc))^{\mathrm{o}} = (\mathrm{tt})(\mathrm{st})$, and:
$$\varepsilon_a := (\lambda f^{\mathrm{s(tt)}}.\ (\lambda h^{\mathrm{tt}}.\ (\lambda i.\ (f(i) = h))));$$

(d_2) if $b = \mathrm{t} \neq c$, then: $(\mathrm{s}b)^{\mathrm{o}} = \mathrm{st}$, $(\mathrm{s}c)^{\mathrm{o}} = c^{\mathrm{o}}(\mathrm{st})$, $(\mathrm{s}(bc))^{\mathrm{o}} = (\mathrm{t}c^{\mathrm{o}})(\mathrm{st})$, and:
$$\varepsilon_a := (\lambda f^{\mathrm{s(t}c)}.\ (\lambda h^{\mathrm{t}c^{\mathrm{o}}}.\ (\lambda i.\ (\forall p^{\mathrm{st}})(\forall D^{\mathrm{s}c}) [(f(i)(p(i)) = D(i)) \leftrightarrow \varepsilon_c(D)(h(p(i))(i)];$$

(d_3) if $b \neq \mathrm{t} = c$, then: $(\mathrm{s}b)^{\mathrm{o}} = b^{\mathrm{o}}(\mathrm{st})$, $(\mathrm{s}c)^{\mathrm{o}} = \mathrm{st}$, $(\mathrm{s}(bc))^{\mathrm{o}} = (b^{\mathrm{o}}\mathrm{t})(\mathrm{st})$, and:
$$\varepsilon_a := (\lambda P^{\mathrm{s}(b\mathrm{t})}.\ (\lambda Q^{b^{\mathrm{o}}\mathrm{t}}.\ (\lambda i. (\forall C^{\mathrm{s}b})[P(i)(C(i)) \leftrightarrow (\exists x^{b^{\mathrm{o}^{\mathrm{o}}}})[\varepsilon_b(C)(x)(i) \wedge Q(x)]])))$$

(d_4) if $b \neq \mathrm{t} \neq c$, then $(\mathrm{s}b)^{\mathrm{o}} = b^{\mathrm{o}}(\mathrm{st})$, $(\mathrm{s}c)^{\mathrm{o}} = c^{\mathrm{o}}(\mathrm{st})$, $(\mathrm{s}(bc))^{\mathrm{o}} = (b^{\mathrm{o}}c^{\mathrm{o}})(\mathrm{st})$, and:
$$\varepsilon_a := (\lambda f^{\mathrm{s}(bc)}.\ (\lambda h^{b^{\mathrm{o}}c^{\mathrm{o}}}.\ (\lambda i.\ (\forall C^{\mathrm{s}b})(\forall D^{\mathrm{s}c}) [(f(i)(C(i)) = D(i)) \leftrightarrow (\exists x^{b^{\mathrm{o}}})[\varepsilon_b(C)(x)(i) \wedge \varepsilon_c(D)(h(x))(i)]])))$$

Now, if $a \in \mathbf{FT} \setminus \{\mathrm{t}\}$, $(\mathrm{s}a)^{\mathrm{o}} = (a^{\mathrm{o}}(\mathrm{st}))$, and so $\dot{\varepsilon}_a$ is a ternary relation between objects of type a ($\neq \mathrm{t}$), their (not necessarily unique) Russellian counterparts, and possible worlds. It can be shown that (C) which counterparts a Fregean intension has for a given world, only depends on its value for that world; that (D) two Fregean intensions share a Russellian counterpart in a given world only if their values for that world coincide; and that (E) at any world, every Fregean intension has at least one such $\dot{\varepsilon}$-counterpart. In other words, the following facts hold for any Fregean type $a \neq \mathrm{t}$:

(C) For any $\boldsymbol{X}_1, \boldsymbol{X}_2 \in D_{\mathrm{s}a}$, and $\boldsymbol{w} \in D_{\mathrm{s}}$:
if $\boldsymbol{X}_1(\boldsymbol{w}) = \boldsymbol{X}_2(\boldsymbol{w})$, then:
$$\{\boldsymbol{z} \in D_{a^{\mathrm{o}}} |\ \dot{\varepsilon}_a(\boldsymbol{X}_1)(\boldsymbol{z})(\boldsymbol{w}) = 1\} = \{\boldsymbol{z} \in D_{a^{\mathrm{o}}} |\ \dot{\varepsilon}_a(\boldsymbol{X}_2)(\boldsymbol{z})(\boldsymbol{w}) = 1\}.$$

(D) For any $\boldsymbol{X}_1, \boldsymbol{X}_2 \in D_{\mathrm{s}a}$, and $\boldsymbol{w} \in D_{\mathrm{s}}$:
if $\boldsymbol{X}_1(\boldsymbol{w}) \neq \boldsymbol{X}_2(\boldsymbol{w})$, then
$$\{\boldsymbol{z} \in D_{a^{\mathrm{o}}} |\ \dot{\varepsilon}_a(\boldsymbol{X}_1)(\boldsymbol{z})(\boldsymbol{w}) = 1\} \cap \{\boldsymbol{z} \in D_{a^{\mathrm{o}}} |\ \dot{\varepsilon}_a(\boldsymbol{X}_2)(\boldsymbol{z})(\boldsymbol{w}) = 1\} = \varnothing.$$

(E) If $\boldsymbol{X} \in D_{\mathrm{s}a}$ and $\boldsymbol{w} \in D_{\mathrm{s}}$, there is a $\boldsymbol{z} \in D_{a^{\mathrm{o}}}$ such that $\dot{\varepsilon}_a(\boldsymbol{X})(\boldsymbol{z})(\boldsymbol{w}) = 1$.

(D) and (E) together imply that $\dot{\varepsilon}_a$ is injective. For if $\boldsymbol{X}_1, \boldsymbol{X}_2 \in D_{\mathrm{s}a}$ and $\boldsymbol{X}_1 \neq \boldsymbol{X}_2$, then $\boldsymbol{X}_1(\boldsymbol{w}_0) \neq \boldsymbol{X}_2(\boldsymbol{w}_0)$, for some $\boldsymbol{w}_0 \in D_{\mathrm{s}}$. Hence, putting $A := \{\boldsymbol{z} \in D_{a^{\mathrm{o}}} |\ \dot{\varepsilon}_a(\boldsymbol{X}_1)(\boldsymbol{z})(\boldsymbol{w}_0) = 1\}$ and $B := \{\boldsymbol{z} \in D_{a^{\mathrm{o}}} |\ \dot{\varepsilon}_a(\boldsymbol{X}_2)(\boldsymbol{z})(\boldsymbol{w}_0) = 1\}$, we have: $A \cap B = \varnothing$, by (D). But then by (E), $A \neq \varnothing$, which means that $A \neq B$ and hence there must be some $\boldsymbol{z}_0 \in D_{a^{\mathrm{o}}}$ such that $\dot{\varepsilon}_a(\boldsymbol{X}_1)(\boldsymbol{w}_0)(\boldsymbol{z}_0) \neq$

$\dot{\varepsilon}_a(\boldsymbol{X}_2)(\boldsymbol{w}_0)(\boldsymbol{z}_0)$, and thus $\dot{\varepsilon}_a(\boldsymbol{X}_1) \neq \dot{\varepsilon}_a(\boldsymbol{X}_2)$. So for each $a \in \mathbf{FT} \smallsetminus \{\mathrm{t}\}$, (D) and (E) give rise to (Inj) – a fact that will come in handy in the proofs below:

(Inj) For any $\boldsymbol{X}, \boldsymbol{Y} \in D_{\mathrm{s}a}$, if $\dot{\varepsilon}_a(\boldsymbol{X}) = \dot{\varepsilon}_a(\boldsymbol{Y})$, then $\boldsymbol{X} = \boldsymbol{Y}$.

Since $\dot{\varepsilon}_{\mathrm{t}}$ is trivially injective, (D), and (E) suffice to establish embeddability of any $(\mathrm{s}a) \in \mathbf{FT}$ into $(\mathrm{s}a)^{\mathrm{o}} \in \mathbf{RT}$. (C), (D), and (E) can be proved by simultaneous recursion, dropping case (a):

$\underline{ad(C)}$: Given $\boldsymbol{X}_1, \boldsymbol{X}_2 \in D_{\mathrm{s}a}$ and $\boldsymbol{w} \in D_{\mathrm{s}}$ such that $\boldsymbol{X}_1(\boldsymbol{w}) = \boldsymbol{X}_2(\boldsymbol{w})$, we show that: for any $\boldsymbol{z} \in D_{a^{\mathrm{o}}}$, $\dot{\varepsilon}_a(\boldsymbol{X}_1)(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff $\dot{\varepsilon}_a(\boldsymbol{X}_2)(\boldsymbol{z})(\boldsymbol{w}) = 1$:

(b) If $\boldsymbol{f}_1(\boldsymbol{w}) = \boldsymbol{f}_2(\boldsymbol{w})$, then by the definition of $\dot{\varepsilon}_{\mathrm{e}}$:
$\dot{\varepsilon}_{\mathrm{e}}(\boldsymbol{f}_1)(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff $\boldsymbol{z} = \boldsymbol{f}_1(\boldsymbol{w})$ iff $\boldsymbol{z} = \boldsymbol{f}_2(\boldsymbol{w})$ iff $\dot{\varepsilon}_{\mathrm{e}}(\boldsymbol{f}_2)(\boldsymbol{w}) = 1$.

(c_1) The same argument applies in this case.

(c_2) If $\boldsymbol{C}_1(\boldsymbol{w}) = \boldsymbol{C}_2(\boldsymbol{w})$, we observe:
$\dot{\varepsilon}_{\mathrm{s}b}(\boldsymbol{C}_1)(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff
$\boldsymbol{z} = \dot{\varepsilon}_b(\boldsymbol{C}_1(\boldsymbol{w}))$ iff — def. $\dot{\varepsilon}_{\mathrm{s}b}$
$\boldsymbol{z} = \dot{\varepsilon}_b(\boldsymbol{C}_2(\boldsymbol{w}))$ iff — assumption
$\dot{\varepsilon}_{\mathrm{s}b}(\boldsymbol{C}_2)(\boldsymbol{z})(\boldsymbol{w}) = 1.$ — def. $\dot{\varepsilon}_{\mathrm{s}b}$

(d_1) The same reasoning as in cases (b) and (c_1) applies.

(d_2) Suppose $\boldsymbol{f}_1(\boldsymbol{w}) = \boldsymbol{f}_2(\boldsymbol{w})$. Then for any $\boldsymbol{z} \in D_{\mathrm{t}c^{\mathrm{o}}}$ we have, by the definition of $\dot{\varepsilon}_{\mathrm{t}c}$:
$\dot{\varepsilon}_{\mathrm{t}c}(\boldsymbol{f}_1)(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff
for all $\boldsymbol{p} \in D_{\mathrm{st}}$ and $\boldsymbol{D} \in D_{\mathrm{s}c}$ the following holds:
$$\boldsymbol{f}_1(\boldsymbol{w})(\boldsymbol{p}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w}) \text{ iff } \dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{p}(\boldsymbol{w})))(\boldsymbol{w}) = 1,$$
which means that, for all $\boldsymbol{p} \in D_{\mathrm{st}}$ and $\boldsymbol{D} \in D_{\mathrm{s}c}$:
$$\boldsymbol{f}_2(\boldsymbol{w})(\boldsymbol{p}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w}) \text{ iff } \dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{p}(\boldsymbol{w})))(\boldsymbol{w}) = 1,$$
given that $\boldsymbol{f}_1(\boldsymbol{w}) = \boldsymbol{f}_2(\boldsymbol{w})$. But then, by the definition of $\dot{\varepsilon}_{\mathrm{t}c}$:
$\dot{\varepsilon}_{\mathrm{t}c}(\boldsymbol{f}_2)(\boldsymbol{z})(\boldsymbol{w}) = 1$.

(d_3) Suppose $\boldsymbol{P}_1(\boldsymbol{w}) = \boldsymbol{P}_2(\boldsymbol{w})$. Then for any $\boldsymbol{z} \in D_{b^{\mathrm{o}}\mathrm{t}}$ we have:
$\dot{\varepsilon}_{b\mathrm{t}}(\boldsymbol{P}_1)(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff
for all $\boldsymbol{C} \in D_{\mathrm{s}b}$ the following holds:
$\boldsymbol{P}_1(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = 1$ iff for some $\boldsymbol{x} \in D_{b^{\mathrm{o}}}, \dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x})(\boldsymbol{w}) = \boldsymbol{z}(\boldsymbol{x}) = 1$.
Hence: $\dot{\varepsilon}_{b\mathrm{t}}(\boldsymbol{P}_1)(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff
for all $\boldsymbol{C} \in D_{\mathrm{s}b}$ the following holds:
$\boldsymbol{P}_2(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = 1$ iff for some $\boldsymbol{x} \in D_{b^{\mathrm{o}}}, \dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x})(\boldsymbol{w}) = \boldsymbol{z}(\boldsymbol{x}) = 1$,
which means that $\dot{\varepsilon}_{b\mathrm{t}}(\boldsymbol{P}_1)(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff $\dot{\varepsilon}_{b\mathrm{t}}(\boldsymbol{P}_2)(\boldsymbol{z})(\boldsymbol{w}) = 1$.

(d_4) Suppose that $\boldsymbol{f}_1(\boldsymbol{w}) = \boldsymbol{f}_2(\boldsymbol{w})$. Then for any $\boldsymbol{z} \in D_{b^\circ c^\circ}$:
$\dot{\varepsilon}_{bc}(\boldsymbol{f}_1)(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff
for all $\boldsymbol{C} \in D_{sb}$ and $\boldsymbol{D} \in D_{sc}$ the following holds:

$$\boldsymbol{f}_1(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = 1 \text{ iff for some } \boldsymbol{x} \in D_{b^\circ}, \dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x})(\boldsymbol{w}) = \dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{x}))(\boldsymbol{w}) = 1.$$

Hence: $\dot{\varepsilon}_{bc}(\boldsymbol{f}_1)(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff
for all $\boldsymbol{C} \in D_{sb}$ and $\boldsymbol{D} \in D_{sc}$ the following holds:

$$\boldsymbol{f}_2(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = 1 \text{ iff for some } \boldsymbol{x} \in D_{b^\circ}, \dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x})(\boldsymbol{w}) = \dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{x}))(\boldsymbol{w}) = 1,$$

which means that $\dot{\varepsilon}_{bc}(\boldsymbol{f}_1)(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff $\dot{\varepsilon}_{bc}(\boldsymbol{f}_2)(\boldsymbol{z})(\boldsymbol{w}) = 1$.

$\underline{ad(D)}$: Given some $\boldsymbol{z} \in D_{a^\circ}$ such that:

(+) $\quad \dot{\varepsilon}_a(\boldsymbol{X}_1)(\boldsymbol{z})(\boldsymbol{w}) = \dot{\varepsilon}_a(\boldsymbol{X}_2)(\boldsymbol{z})(\boldsymbol{w}) = 1,$

we show that $\boldsymbol{X}_1(\boldsymbol{w}) = \boldsymbol{X}_2(\boldsymbol{w})$:

(b) By the definition of $\dot{\varepsilon}_{\mathrm{e}}$, (+) implies that $\boldsymbol{z} = \boldsymbol{X}_1(\boldsymbol{w}) = \boldsymbol{X}_2(\boldsymbol{w})$.

(c_1) The same argument applies as in the previous case.

(c_2) By the definition of $\dot{\varepsilon}_{\mathrm{s}b}$, (+) implies that $\boldsymbol{z} = \dot{\varepsilon}_b(\boldsymbol{X}_1(\boldsymbol{w})) = \dot{\varepsilon}_b(\boldsymbol{X}_2(\boldsymbol{w}))$. But then $\boldsymbol{X}_1(\boldsymbol{w}) = \boldsymbol{X}_2(\boldsymbol{w})$, because b satisfies (*Inj*), by induction.

(d_1) The argument from case (b) applies again.

(d_2) By the definition of $\dot{\varepsilon}_{\mathrm{t}c}$, (+) implies that for any $\boldsymbol{p} \in D_{\mathrm{st}}$ and $\boldsymbol{D} \in D_{\mathrm{s}c}$:
$\boldsymbol{X}_1(\boldsymbol{w})(\boldsymbol{p}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$ iff $\dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{p}(\boldsymbol{w}))(\boldsymbol{w}) = 1$ and:
$\boldsymbol{X}_2(\boldsymbol{w})(\boldsymbol{p}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$ iff $\dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{p}(\boldsymbol{w}))(\boldsymbol{w}) = 1$.
So, for any such $\boldsymbol{p}$ and $\boldsymbol{D}$ we have:
$\boldsymbol{X}_1(\boldsymbol{w})(\boldsymbol{p}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$ iff $\boldsymbol{X}_2(\boldsymbol{w})(\boldsymbol{p}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$.
In particular, for any $n \in D_{\mathrm{t}}$ and $\boldsymbol{y} \in D_c$, we may consider the constant functions $\boldsymbol{p}_n := \{(\boldsymbol{w}', n) | \boldsymbol{w}' \in D_{\mathrm{s}}\} \in D_{\mathrm{st}}$ and $\boldsymbol{D}_{\boldsymbol{y}} := \{(\boldsymbol{w}', \boldsymbol{y}) | \boldsymbol{w}' \in D_{\mathrm{s}}\} \in D_{\mathrm{s}c}$ to conclude:
$\boldsymbol{X}_1(\boldsymbol{w})(\boldsymbol{p}_n(\boldsymbol{w})) = \boldsymbol{D}_{\boldsymbol{y}}(\boldsymbol{w})$ iff $\boldsymbol{X}_2(\boldsymbol{w})(\boldsymbol{p}_n(\boldsymbol{w})) = \boldsymbol{D}_{\boldsymbol{y}}(\boldsymbol{w})$, i.e.:
$\boldsymbol{X}_1(\boldsymbol{w})(n) = \boldsymbol{y}$ iff $\boldsymbol{X}_2(\boldsymbol{w})(n) = \boldsymbol{y}$, for any $n \in D_{\mathrm{t}}$ and $\boldsymbol{y} \in D_c$, which means that $\boldsymbol{X}_1(\boldsymbol{w}) = \boldsymbol{X}_2(\boldsymbol{w})$, by extensionality.

(d_3) By the definition of $\dot{\varepsilon}_{b\mathrm{t}}$, (+) implies that for any $\boldsymbol{C} \in D_{\mathrm{s}b}$:
$\boldsymbol{X}_1(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = 1$ iff
there is an $\boldsymbol{x}_1 \in D_{b^\circ}$ such that: $\dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x}_1)(\boldsymbol{w}) = \boldsymbol{z}(\boldsymbol{x}_1) = 1$ and:
$\boldsymbol{X}_2(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = 1$ iff
there is an $\boldsymbol{x}_2 \in D_{b^\circ}$ such that: $\dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x}_2)(\boldsymbol{w}) = \boldsymbol{z}(\boldsymbol{x}) = 1$.
So, for any such $\boldsymbol{D}$ we have:
$\boldsymbol{X}_1(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = 1$ iff $\boldsymbol{X}_2(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = 1$.
As in the previous case, we may now consider the constant functions $\boldsymbol{D}_{\boldsymbol{y}}$ (where $\boldsymbol{y} \in D_b$) to conclude:
$\boldsymbol{X}_1(\boldsymbol{w})(\boldsymbol{C}_{\boldsymbol{y}}(\boldsymbol{w})) = 1$ iff $\boldsymbol{X}_2(\boldsymbol{w})(\boldsymbol{C}_{\boldsymbol{y}}(\boldsymbol{w})) = 1$, i.e.:

$\boldsymbol{X}_1(\boldsymbol{w})(\boldsymbol{y}) = 1$ iff $\boldsymbol{X}_2(\boldsymbol{w})(\boldsymbol{y}) = 1$, for any $\boldsymbol{y} \in D_b$, and thus:
$\boldsymbol{X}_1(\boldsymbol{w}) = \boldsymbol{X}_2(\boldsymbol{w})$, by extensionality.

(d_4) By the definition of $\dot{\varepsilon}_{\mathrm{t}c}$, (+) implies that for any $\boldsymbol{C} \in D_{\mathrm{s}b}$ and $\boldsymbol{D} \in D_{\mathrm{s}c}$:
$\boldsymbol{X}_1(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$ iff
for some $\boldsymbol{x}_1 \in D_{b^\circ}$: $\dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x}_1)(\boldsymbol{w}) = \dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{x}_1))(\boldsymbol{w}) = 1$ and:
$\boldsymbol{X}_2(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$ iff
for some $\boldsymbol{x}_2 \in D_{b^\circ}$: $\dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x}_2)(\boldsymbol{w}) = \dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{x}_2))(\boldsymbol{w}) = 1$.
So we have, for any such $\boldsymbol{C}$ and $\boldsymbol{D}$:
$\boldsymbol{X}_1(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$ iff $\boldsymbol{X}_2(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$.
As before, we now invoke constant functions $\boldsymbol{C}_{\boldsymbol{x}}$ and $\boldsymbol{D}_{\boldsymbol{y}}$ (where $\boldsymbol{x} \in D_b$ and $\boldsymbol{y} \in D_c$) to conclude:
$\boldsymbol{X}_1(\boldsymbol{w})(\boldsymbol{C}_{\boldsymbol{x}}(\boldsymbol{w})) = \boldsymbol{D}_{\boldsymbol{y}}(\boldsymbol{w})$ iff $\boldsymbol{X}_2(\boldsymbol{w})(\boldsymbol{C}_{\boldsymbol{x}}(\boldsymbol{w})) = \boldsymbol{D}_{\boldsymbol{y}}(\boldsymbol{w})$, i.e.:
$\boldsymbol{X}_1(\boldsymbol{w})(\boldsymbol{x}) = \boldsymbol{y}$ iff $\boldsymbol{X}_2(\boldsymbol{w})(\boldsymbol{x}) = \boldsymbol{y}$, for any $\boldsymbol{x} \in D_c$ and $\boldsymbol{y} \in D_d$,
which means that $\boldsymbol{X}_1(\boldsymbol{w}) = \boldsymbol{X}_2(\boldsymbol{w})$, by extensionality.

ad(E):

(b) Given $\boldsymbol{f} \in D_{\mathrm{se}}$ and $\boldsymbol{w} \in D_{\mathrm{s}}$, we put $\boldsymbol{z} := \boldsymbol{f}(\boldsymbol{w})$ and observe that: $\dot{\varepsilon}_{\mathrm{e}}(\boldsymbol{f})(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff $\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{w})$, which is true by definition.

(c_1) Analogously, wet put $\boldsymbol{z} := \boldsymbol{C}(\boldsymbol{w})$, for any $\boldsymbol{C} \in D_{\mathrm{s(st)}}$ and $\boldsymbol{w} \in D_{\mathrm{s}}$.

(c_2) In a similar vein, given $\boldsymbol{C} \in D_{\mathrm{s(s}b)}$ and $\boldsymbol{w} \in D_{\mathrm{s}}$, $\boldsymbol{z} := \dot{\varepsilon}_b(\boldsymbol{C}(\boldsymbol{w}))$ fits the bill (and uniquely so).

(d_1) As in cases (b) and (c_1), we put: $\boldsymbol{z} := \boldsymbol{f}(\boldsymbol{w})$, whenever $\boldsymbol{f} \in D_{\mathrm{s(tt)}}$ and $\boldsymbol{w} \in D_{\mathrm{s}}$.

(d_2) Given $\boldsymbol{f} \in D_{\mathrm{s(t}c)}$ and $\boldsymbol{w} \in D_{\mathrm{s}}$, we first apply the inductive hypothesis to c to observe that, for any $\boldsymbol{D} \in D_{\mathrm{s}b}$:

$$\boldsymbol{Y}_{\boldsymbol{D}} := \{\boldsymbol{y} \in D_{c^\circ} |\ \dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{y})(\boldsymbol{w}) = 1\} \neq \varnothing.$$

So we can select a fixed object $\boldsymbol{y}_{\boldsymbol{D}} \in D_{c^\circ}$ from each $\boldsymbol{Y}_{\boldsymbol{D}}$ and put:
$\boldsymbol{z} := \{(\boldsymbol{p}(\boldsymbol{w}), \boldsymbol{y}_{\boldsymbol{D}}) |\ \boldsymbol{p} \in D_{\mathrm{st}}, \boldsymbol{D} \in D_{\mathrm{s}c}, \text{ and } \boldsymbol{f}(\boldsymbol{w})(\boldsymbol{p}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})\}$[xiv]
We first need to show that $\boldsymbol{z} \in D_{\mathrm{t}c^\circ}$. So suppose $n \in D_{\mathrm{t}}$. Then, putting

$$\boldsymbol{p}[n] := \{(\boldsymbol{w}, n) |\ \boldsymbol{w} \in D_{\mathrm{s}}\} \in D_{\mathrm{st}} \text{ and}$$
$$\boldsymbol{D}[n] := \{(\boldsymbol{w}', \boldsymbol{f}(\boldsymbol{w}')(n)) |\ \boldsymbol{w}' \in D_{\mathrm{s}}\} \in D_{\mathrm{s}c},$$

we observe that $\boldsymbol{D}[n](\boldsymbol{w}) = \boldsymbol{f}(\boldsymbol{w})(n) = \boldsymbol{f}(\boldsymbol{w})(\boldsymbol{p}[n](\boldsymbol{w}))$, and thus: $(n, \boldsymbol{y}_{\boldsymbol{D}[n]}) \in \boldsymbol{z}$. So $dom(\boldsymbol{z}) = D_{\mathrm{t}}$. As to uniqueness, suppose that $(n, \boldsymbol{y}), (n, \boldsymbol{y}') \in \boldsymbol{z}$. Then, for some $\boldsymbol{p}, \boldsymbol{p}' \in D_{\mathrm{st}}$ and $\boldsymbol{D}, \boldsymbol{D}' \in D_{\mathrm{s}c}$:

[xiv] To be sure, this is short for:
$\{x |$ there are $n \in D_{\mathrm{t}}, \boldsymbol{y} \in D_{c^\circ}, \boldsymbol{p} \in D_{\mathrm{st}}$ and $\boldsymbol{D} \in D_{\mathrm{s}c}$ such that:
$x = (n, \boldsymbol{y}), n = \boldsymbol{p}(\boldsymbol{w}), \boldsymbol{y} = \boldsymbol{y}_{\boldsymbol{D}}$, and $\boldsymbol{f}(\boldsymbol{w})(n) = \boldsymbol{D}(\boldsymbol{w})\}$.

(i) $\boldsymbol{p}(\boldsymbol{w}) = \boldsymbol{p}'(\boldsymbol{w}) = n$; and

(ii) $\boldsymbol{D}(\boldsymbol{w}) = \boldsymbol{f}(\boldsymbol{w})(n) = \boldsymbol{D}'(\boldsymbol{w})$;

(iii) $\boldsymbol{y} = \boldsymbol{y_D}$ and $\boldsymbol{y}' = \boldsymbol{y_{D'}}$.

By induction, (D) applies to c and so, (ii) implies: $\boldsymbol{Y_D} = \boldsymbol{Y_{D'}}$, which means that $\boldsymbol{y_D} = \boldsymbol{y_{D'}}$, and so: $\boldsymbol{y} = \boldsymbol{y_D} = \boldsymbol{y_{D'}} = \boldsymbol{y}'$, by ($iii$).

Now we know that $\boldsymbol{z} \in D_{\mathrm{tc}^\circ}$, we can apply $\dot{\varepsilon}_{\mathrm{tc}}$ to conclude:
$\dot{\varepsilon}_{\mathrm{tc}}(\boldsymbol{f})(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff for all $\boldsymbol{p} \in D_{\mathrm{st}}$ and $\boldsymbol{D} \in D_{\mathrm{sc}}$:
(!) $\boldsymbol{f}(\boldsymbol{w})(\boldsymbol{p}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$ iff $\dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{p}(\boldsymbol{w})))(\boldsymbol{w}) = 1$.
And, indeed, given any $\boldsymbol{p} \in D_{\mathrm{st}}$ and $\boldsymbol{D} \in D_{\mathrm{sc}}$, we can show that they satisfy both directions of (!):
"$\Rightarrow$": Suppose $\boldsymbol{f}(\boldsymbol{w})(\boldsymbol{p}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$. Hence $\boldsymbol{z}(\boldsymbol{p}(\boldsymbol{w})) = \boldsymbol{y_D}$, by the definition of $\boldsymbol{z}$. But $\boldsymbol{y_D} \in \boldsymbol{Y_D}$, and so: $\dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{p}(\boldsymbol{w})))(\boldsymbol{w}) = \dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{y_D})(\boldsymbol{w}) = 1$.
'$\Leftarrow$": Suppose that $\dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{p}(\boldsymbol{w})))(\boldsymbol{w}) = 1$. Then $\boldsymbol{z}(\boldsymbol{p}(\boldsymbol{w})) \in \boldsymbol{Y_D}$. Now, by the definition of $\boldsymbol{z}$, $\boldsymbol{z}(\boldsymbol{p}(\boldsymbol{w})) = \boldsymbol{y_{D'}} \in \boldsymbol{Y_{D'}}$, for some $\boldsymbol{D}' \in D_{\mathrm{sc}}$, (#) $\boldsymbol{f}(\boldsymbol{w})(\boldsymbol{p}(\boldsymbol{w})) = \boldsymbol{D}'(w)$. But then $\boldsymbol{z}(\boldsymbol{p}(\boldsymbol{w})) \in \boldsymbol{Y_D} \cap \boldsymbol{Y_{D'}} \neq \varnothing$, which means that $\boldsymbol{D}(\boldsymbol{w}) = \boldsymbol{D}'(\boldsymbol{w})$, by ($D$), inductively applied to c. So by (#), we conclude that $\boldsymbol{f}(\boldsymbol{w})(\boldsymbol{p}(\boldsymbol{w})) = \boldsymbol{D}(w)$

(d_3) Given $\boldsymbol{P} \in D_{\mathrm{s}(bt)}$ and $\boldsymbol{w} \in D_{\mathrm{s}}$, we put
$\boldsymbol{z} :=$ "$\{\boldsymbol{x} \in D_{b^\circ} |$ for some $\boldsymbol{D} \in D_{\mathrm{s}b}$: $\boldsymbol{P}(\boldsymbol{w})(\boldsymbol{D}(\boldsymbol{w})) = \dot{\varepsilon}_b(\boldsymbol{D})(\boldsymbol{x})(\boldsymbol{w}) = 1\}$".
We then have: $\dot{\varepsilon}_{bt}(\boldsymbol{P})(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff for all $\boldsymbol{C} \in D_{\mathrm{s}b}$:
(!!) $\boldsymbol{P}(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = 1$ iff for some $\boldsymbol{x} \in D_{b^\circ}$: $\dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x})(\boldsymbol{w}) = \boldsymbol{z}(\boldsymbol{x}) = 1$.
And, indeed, given any $\boldsymbol{C} \in D_{\mathrm{s}b}$, we can show that it satisfies both directions of (!!):
"$\Rightarrow$": Suppose $\boldsymbol{P}(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = 1$. By induction, there is an $\boldsymbol{x} \in D_{b^\circ}$ such that $\dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x})(\boldsymbol{w}) = 1$ and thus $\boldsymbol{x}$ trivially satisfies: $\boldsymbol{z}(\boldsymbol{x}) = 1$, due to the definition of $\boldsymbol{z}$.
"$\Leftarrow$": Suppose $\boldsymbol{x} \in D_{b^\circ}$ and $\dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x})(\boldsymbol{w}) = \boldsymbol{z}(\boldsymbol{x}) = 1$. Then, by the definition of $\boldsymbol{z}$: $\dot{\varepsilon}_b(\boldsymbol{D})(\boldsymbol{x})(\boldsymbol{w}) = 1$, for some $\boldsymbol{D} \in D_{\mathrm{s}b}$ such that $\boldsymbol{P}(\boldsymbol{w})(\boldsymbol{D}(\boldsymbol{w})) = 1$. Now, by induction, ($D$) applies to b, and so $\boldsymbol{C}(\boldsymbol{w}) = \boldsymbol{D}(\boldsymbol{w})$, given that $\dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x})(\boldsymbol{w}) = \dot{\varepsilon}_b(\boldsymbol{D})(\boldsymbol{x})(\boldsymbol{w}) = 1$. But then we also have: $\boldsymbol{P}(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = 1$, given that $\boldsymbol{P}(\boldsymbol{w})(\boldsymbol{D}(\boldsymbol{w})) = 1$.

(d_4) Given $\boldsymbol{f} \in D_{\mathrm{s}(bc)}$ and $\boldsymbol{w} \in D_{\mathrm{s}}$, we first apply the inductive hypothesis to b and c to observe that, for any $\boldsymbol{C} \in D_{\mathrm{s}b}$ and $\boldsymbol{D} \in D_{\mathrm{sb}}$:

$$\boldsymbol{X_C} := \{\boldsymbol{x} \in D_{b^\circ} | \dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x})(\boldsymbol{w}) = 1\} \neq \varnothing \text{ and:}$$
$$\boldsymbol{Y_D} := \{\boldsymbol{y} \in D_{c^\circ} | \dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{y})(\boldsymbol{w}) = 1\} \neq \varnothing.$$

So we can select a fixed object $\boldsymbol{y_D} \in D_{c^\circ}$ from each $\boldsymbol{Y_D}$ and put:

$\boldsymbol{z} := \{(\boldsymbol{x},\boldsymbol{y}) \in D_{b^\circ} \times D_{c^\circ} |$ there are $\boldsymbol{C} \in D_{sb}$ and $\boldsymbol{D} \in D_{\mathrm{sc}}$ such that:
$$\boldsymbol{x} \in \boldsymbol{X_C},\ \boldsymbol{f}(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w}), \text{ and } \boldsymbol{y} = \boldsymbol{y_D}\}$$
$$\cup \{(\boldsymbol{x},\boldsymbol{y}_0) |\ \boldsymbol{x} \in D_{b^\circ} \smallsetminus \textstyle\bigcup_{\boldsymbol{C} \in D_{sb}} \boldsymbol{X_C}\},$$
where $\boldsymbol{y}_0 \in D_{c^\circ}$ is some fixed object. We first need to show that $\boldsymbol{z} \in D_{b^\circ c^\circ}$. So suppose $\boldsymbol{x} \in D_{b^\circ}$. Then either (Case 1) $\boldsymbol{x} \notin \boldsymbol{X_C}$, for any $\boldsymbol{C} \in D_{sb}$, and thus $(\boldsymbol{x},\boldsymbol{y}_0) \in \boldsymbol{z}$. Or else (Case 2) $\boldsymbol{x} \in \boldsymbol{X_C}$, where $\boldsymbol{C} \in D_{sb}$; then, putting $\boldsymbol{D} := \{(\boldsymbol{w}', \boldsymbol{f}(\boldsymbol{w}')(\boldsymbol{C}(\boldsymbol{w}'))) |\ \boldsymbol{w}' \in D_{\mathrm{s}}\} \in D_{sc}$, we observe that $\boldsymbol{f}(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$ and thus $(\boldsymbol{x},\boldsymbol{y_D}) \in \boldsymbol{z}$. So $dom(\boldsymbol{z}) = D_{b^\circ}$. As to uniqueness, suppose that $(\boldsymbol{x},\boldsymbol{y}),(\boldsymbol{x},\boldsymbol{y}') \in \boldsymbol{z}$. Then if Case 1 holds, $\boldsymbol{y} = \boldsymbol{y}_0 = \boldsymbol{y}'$. And if (Case 2) holds, there are $\boldsymbol{C},\boldsymbol{C}' \in D_{sb}$ and $\boldsymbol{D},\boldsymbol{D}' \in D_{\mathrm{sc}}$ such that:

(*i*) $\boldsymbol{x} \in \boldsymbol{X_C}$ and $\boldsymbol{x} \in \boldsymbol{X_{C'}}$; and

(*ii*) $\boldsymbol{f}(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$ and $\boldsymbol{f}(\boldsymbol{w})(\boldsymbol{C}'(\boldsymbol{w})) = \boldsymbol{D}'(\boldsymbol{w})$;

(*iii*) $\boldsymbol{y} = \boldsymbol{y_D}$ and $\boldsymbol{y}' = \boldsymbol{y_{D'}}$.

By induction, (D) applies to b and so, (i) implies: $\boldsymbol{C}(\boldsymbol{w}) = \boldsymbol{C}'(\boldsymbol{w})$. But then, by (ii): $\boldsymbol{D}(\boldsymbol{w}) = \boldsymbol{D}'(\boldsymbol{w})$. So, applying (D) to c, we find: $\boldsymbol{Y_D} = \boldsymbol{Y_{D'}}$, which means that $\boldsymbol{y_D} = \boldsymbol{y_{D'}}$, and so: $\boldsymbol{y} = \boldsymbol{y_D} = \boldsymbol{y_{D'}} = \boldsymbol{y}'$, by (iii).

Now we know that $\boldsymbol{z} \in D_{b^\circ c^\circ}$, we can apply $\dot{\varepsilon}_{bc}$ and conclude:
$\dot{\varepsilon}_{bc}(\boldsymbol{f})(\boldsymbol{z})(\boldsymbol{w}) = 1$ iff for all $\boldsymbol{C} \in D_{sb}$ and $\boldsymbol{D} \in D_{\mathrm{sc}}$:
(!!!) $\boldsymbol{f}(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$ iff for some $\boldsymbol{x} \in D_{b^\circ}$:
$$\dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x})(\boldsymbol{w}) = \dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{x}))(\boldsymbol{w}) = 1.$$
And, indeed, given any $\boldsymbol{C} \in D_{sb}$ and $\boldsymbol{D} \in D_{\mathrm{sc}}$, we can show that they satisfy both directions of (!!!):
"⇒": Suppose $\boldsymbol{f}(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$. By induction there is some $\boldsymbol{x} \in D_{b^\circ}$ such that $\dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x})(\boldsymbol{w}) = 1$, i.e.: $\boldsymbol{x} \in \boldsymbol{X_C}$, and thus, by the definition of $\boldsymbol{z}$: $\boldsymbol{z}(\boldsymbol{x}) = \boldsymbol{y_D} \in \boldsymbol{Y_D}$, whence $\dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{x}))(\boldsymbol{w}) = \dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{y_D})(\boldsymbol{w}) = 1$.

'⇐": Suppose that $\dot{\varepsilon}_b(\boldsymbol{C})(\boldsymbol{x})(\boldsymbol{w}) = \dot{\varepsilon}_c(\boldsymbol{D})(\boldsymbol{z}(\boldsymbol{x}))(\boldsymbol{w}) = 1$. Then $\boldsymbol{x} \in \boldsymbol{X_C}$ and thus, for some $\boldsymbol{D}' \in D_{\mathrm{sc}}$, we have: $\boldsymbol{f}(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = \boldsymbol{D}'(\boldsymbol{w})$ and $\boldsymbol{z}(\boldsymbol{x}) = \boldsymbol{y_{D'}}$. But then $\boldsymbol{z}(\boldsymbol{x}) \in \boldsymbol{Y_D} \cap \boldsymbol{Y_{D'}}$ and, inductively applying (D) to c, we may conclude that $\boldsymbol{D}(\boldsymbol{w}) = \boldsymbol{D}'(\boldsymbol{w})$ and thus that $\boldsymbol{f}(\boldsymbol{w})(\boldsymbol{C}(\boldsymbol{w})) = \boldsymbol{D}(\boldsymbol{w})$.

A.6 Translation Theorem

To be shown: $[\![\overline{\alpha}]\!]^{F^*,g^*} = \dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ \alpha)]\!]^{F,g})$, for any $\alpha \in IL^*_a$ (where $a \in \mathbf{FT}$). The proof proceeds by induction on α where, without loss of generality, we will rely on the paraphrases given in observation **o)** above:

– If $\alpha = \mathrm{c}(i)$, where $\mathrm{c} \in Con_{\mathrm{sa}}$, then:

$$\begin{array}{lll} & [\![\overline{\alpha}]\!]^{F^*,g^*} & \\ = & [\![\tilde{\mathrm{c}}]\!]^{F^*,g^*} & \text{definition of } \overline{\alpha} \\ = & F^*(\tilde{\mathrm{c}}) & \text{interpretation of } IL^* \\ = & \dot{\kappa}_{\mathrm{sa}}(F(\mathrm{c})) & \text{definition of } F^* \\ = & \dot{\kappa}_{\mathrm{sa}}([\hat{\boldsymbol{w}} : F(\mathrm{c})(\boldsymbol{w})]) & \eta\text{-conversion} \\ = & \dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\alpha)]\!]^{F,g}) & \text{interpretation of } IL^* \end{array}$$

– If $\alpha = x \in Var_a$, we first show that:

(!) $[\![(\lambda C^{(\mathrm{sa})^*}. [\Box\ \Psi_a(C)(\tilde{x})])]\!]^{F^*,g^*} = /\dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ \alpha)]\!]^{F,g})/$.

Indeed, for any $\boldsymbol{C} \in D_{(\mathrm{sa})^*}$, we have:

$$\begin{array}{lll} & [\![(\lambda C^{(\mathrm{sa})^*}. [\Box\ \Psi_a(C)(\tilde{x})])]\!]^{F^*,g^*}(\boldsymbol{C}) = 1 & \\ \text{iff} & \text{for all } \boldsymbol{w} \in D_{\mathrm{s}}, [\![\Psi_a]\!](\boldsymbol{C})(g^*(\tilde{x}))(\boldsymbol{w}) = 1 & \text{interpretation of } IL^* \\ \text{iff} & \text{for all } \boldsymbol{w} \in D_{\mathrm{s}} \text{ there is a } \boldsymbol{c} \in D_{\mathrm{sa}} \text{ such that:} & \\ & \boldsymbol{C} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{c}) \text{ and } g^*(\tilde{x}) = \dot{\kappa}_a(\boldsymbol{c}(\boldsymbol{w})) & \Psi\text{-Lemma} \\ \text{iff} & \text{for all } \boldsymbol{w} \in D_{\mathrm{s}} \text{ there is a } \boldsymbol{c} \in D_{\mathrm{sa}} \text{ such that:} & \\ & \boldsymbol{C} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{c}) \text{ and } \dot{\kappa}_a(g(x)) = \dot{\kappa}_a(\boldsymbol{c}(\boldsymbol{w})) & \text{def. } g^* \\ \text{iff} & \text{for all } \boldsymbol{w} \in D_{\mathrm{s}} \text{ there is a } \boldsymbol{c} \in D_{\mathrm{sa}} \text{ such that:} & \\ & \dot{\kappa}_{\mathrm{sa}}^{-1}(\boldsymbol{C}) = \boldsymbol{c} \text{ and } g(x) = \boldsymbol{c}(\boldsymbol{w}) & \text{injectivity of } \dot{\kappa}_{\mathrm{sa}} \text{ and } \dot{\kappa}_a \\ \text{iff} & \text{for all } \boldsymbol{w} \in D_{\mathrm{s}}\text{: } g(x) = \dot{\kappa}_{\mathrm{sa}}^{-1}(\boldsymbol{C})(\boldsymbol{w}) & \text{identity law} \\ \text{iff} & \text{for all } \boldsymbol{w} \in D_{\mathrm{s}}\text{: } [\hat{\boldsymbol{w}} : g(x)](\boldsymbol{w}) = \dot{\kappa}_{\mathrm{sa}}^{-1}(\boldsymbol{C})(\boldsymbol{w}) & \beta\text{-conversion} \\ \text{iff} & [\hat{\boldsymbol{w}}\text{: } g(x)] = \dot{\kappa}_{\mathrm{sa}}^{-1}(\boldsymbol{C}) & \text{extensionality} \\ \text{iff} & \dot{\kappa}_{\mathrm{sa}}([\hat{\boldsymbol{w}}\text{: } g(x)]) = \boldsymbol{C} & \text{def. } \dot{\kappa}_{\mathrm{sa}}^{-1} \\ \text{iff} & \dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ x)]\!]^{F,g}) = \boldsymbol{C} & IL^*\text{-interpretation} \end{array}$$

Given (!), we thus have:

$[\![\overline{\alpha}]\!]^{F^*,g^*} = \dot{\sqcap}_{(\mathrm{sa})^*}(/\dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ \alpha)]\!]^{F,g})/)$.

But then $(\mathrm{s}a)^*$ is of the form $(\underset{\rightarrow}{a}(\mathrm{st}))$, by **i)**, and so **q)** applies and we may conclude that:

$[\![\overline{\alpha}]\!]^{F^*,g^*} = \dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ \alpha)]\!]^{F,g})$.

– If $\alpha = (\beta = \gamma)$, where $\beta, \gamma \in IL_b^*$, it needs to be shown that:

$[\![\overline{(\beta = \gamma)}]\!]^{F^*,g^*} = [\![(\lambda i.(\beta = \gamma))]\!]^{F,g}$,

since $\dot{\kappa}_{\mathrm{st}} = \boldsymbol{id}_{\mathrm{st}}$. And indeed, for any $\boldsymbol{w} \in D_{\mathrm{s}}$ we have:

$$\begin{array}{lll} & [\![\overline{(\beta = \gamma)}]\!]^{F^*,g^*}(\boldsymbol{w}) = 1 & \\ \text{iff} & \text{there is a } \boldsymbol{Z} \in D_{b^*} \text{ such that:} & \\ & [\![\Psi_b]\!]([\![\overline{\beta}]\!]^{F^*,g^*})(\boldsymbol{Z})(\boldsymbol{w}) = 1 \text{ and } [\![\Psi_b]\!]([\![\overline{\gamma}]\!]^{F^*,g^*})(\boldsymbol{Z})(\boldsymbol{w}) = 1 & IL^*\text{-int.} \\ \text{iff} & \text{there is a } \boldsymbol{Z} \in D_{b^*} \text{ such that:} & \\ & [\![\Psi_b]\!](\dot{\kappa}_{\mathrm{s}b}([\![(\lambda i.\ \beta)]\!]^{F,g})(\boldsymbol{Z})(\boldsymbol{w}) = [\![\Psi_b]\!](\dot{\kappa}_{\mathrm{s}b}([\![(\lambda i.\ \gamma)]\!]^{F,g})(\boldsymbol{Z})(\boldsymbol{w}) = 1 & \text{ind'n} \\ \text{iff} & \text{there are } \boldsymbol{c}, \boldsymbol{d} \in D_{\mathrm{s}b} \text{ such that:} & \\ & \dot{\kappa}_{\mathrm{s}b}([\![(\lambda i.\ \beta)]\!]^{F,g}) = \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{c}), \dot{\kappa}_{\mathrm{s}b}([\![(\lambda i.\ \gamma)]\!]^{F,g}) = \dot{\kappa}_{\mathrm{s}b}(\boldsymbol{d}), & \\ & \text{and } \dot{\kappa}_b(\boldsymbol{c}(\boldsymbol{w})) = \dot{\kappa}_b(\boldsymbol{d}(\boldsymbol{w})) & \Psi\text{-Lemma} \\ \text{iff} & \text{there are } \boldsymbol{c}, \boldsymbol{d} \in D_{\mathrm{s}b} \text{ such that:} & \\ & [\![(\lambda i.\ \beta)]\!]^{F,g} = \boldsymbol{c}, [\![(\lambda i.\ \gamma)]\!]^{F,g} = \boldsymbol{d} \text{ and } \boldsymbol{c}(\boldsymbol{w}) = \boldsymbol{d}(\boldsymbol{w}) & \text{injectivity of } \dot{\kappa}_b \text{ and } \dot{\kappa}_{\mathrm{s}b} \end{array}$$

iff $\llbracket(\lambda i.\ \beta)\rrbracket^{F,g}(\boldsymbol{w}) = \llbracket(\lambda i.\ \gamma)\rrbracket^{F,g}(\boldsymbol{w})$ identity

iff $\llbracket\beta\rrbracket^{F,g[^i/\boldsymbol{w}]} = \llbracket\gamma\rrbracket^{F,g[^i/\boldsymbol{w}]}$ IL^*-interpretation

iff $\llbracket(\beta = \gamma)\rrbracket^{F,g[^i/\boldsymbol{w}]} = 1$ IL^*-interpretation

iff $\llbracket(\lambda i.\ (\beta = \gamma))\rrbracket^{F,g}(\boldsymbol{w}) = 1$ IL^*-interpretation

– If $\alpha = \beta(\gamma)$, where $\beta \in IL^*_{ba}$, $\gamma \in IL^*_b$ and $b \in \mathbf{FT}$, then $\overline{\alpha}$ is of the form $\sqcap_{(\mathrm{sa})^*}(\delta)$, where $\delta \in IL^*_{(\mathrm{sa})^*\mathrm{t}}$. We first show that:

(!!) $\llbracket\delta\rrbracket^{F^*,g^*} = /\dot\kappa_{\mathrm{sa}}(\llbracket(\lambda i.\ \beta(\gamma))\rrbracket^{F,g})/$.

Indeed, for any $\boldsymbol{C} \in D_{(\mathrm{sa})^*}$, we have:

$\llbracket\delta\rrbracket^{F^*,g^*}(\boldsymbol{C}) = 1$

iff for all $\boldsymbol{F} \in D_{(ba)^*}$, $\boldsymbol{X} \in D_{b^*}$ and $\boldsymbol{w} \in D_{\mathrm{s}}$ such that
$\llbracket\Psi_{ba}\rrbracket(\llbracket\overline{\beta}\rrbracket^{F^*,g^*})(\boldsymbol{F})(\boldsymbol{w}) = \llbracket\Psi_b\rrbracket(\llbracket\overline{\gamma}\rrbracket^{F^*,g^*})(\boldsymbol{X})(\boldsymbol{w}) = 1$,
there is a $\boldsymbol{Y} \in D_{a^*}$ such that:
$\llbracket\Psi_a\rrbracket(\boldsymbol{C})(\boldsymbol{Y})(\boldsymbol{w}) = \llbracket\Phi_{ba}\rrbracket(\boldsymbol{F})(\boldsymbol{X})(\boldsymbol{Y}) = 1$ IL^*-interpretation

iff for all $\boldsymbol{F} \in D_{(ba)^*}$, $\boldsymbol{X} \in D_{b^*}$ and $\boldsymbol{w} \in D_{\mathrm{s}}$ such that
$\llbracket\Psi_{ba}\rrbracket(\dot\kappa_{\mathrm{s}(ba)}(\llbracket(\lambda i.\ \beta)\rrbracket^{F,g}))(\boldsymbol{F})(\boldsymbol{w}) = \llbracket\Psi_b\rrbracket(\dot\kappa_{\mathrm{s}b}(\llbracket(\lambda i.\ \gamma)\rrbracket^{F,g}))(\boldsymbol{X})(\boldsymbol{w}) = 1$,
there is a $\boldsymbol{Y} \in D_{a^*}$ such that:
$\llbracket\Psi_a\rrbracket(\boldsymbol{C})(\boldsymbol{Y})(\boldsymbol{w}) = \llbracket\Phi_{ba}\rrbracket(\boldsymbol{F})(\boldsymbol{X})(\boldsymbol{Y}) = 1$ induction

iff for all $\boldsymbol{F} \in D_{(ba)^*}$, $\boldsymbol{X} \in D_{b^*}, \boldsymbol{w} \in D_{\mathrm{s}}, \boldsymbol{h} \in D_{\mathrm{s}(ba)}$ and $\boldsymbol{c} \in D_{\mathrm{s}b}$
such that $\dot\kappa_{\mathrm{s}(ba)}(\llbracket(\lambda i.\ \beta)\rrbracket^{F,g}) = \dot\kappa_{\mathrm{s}(ba)}(\boldsymbol{h}), \boldsymbol{F} = \dot\kappa_{ba}(\boldsymbol{h}(\boldsymbol{w}))$,
$\dot\kappa_{\mathrm{s}b}(\llbracket(\lambda i.\ \gamma)\rrbracket^{F,g}) = \dot\kappa_{\mathrm{s}b}(\boldsymbol{c})$, and $\boldsymbol{X} = \dot\kappa_b\boldsymbol{c}(\boldsymbol{w}))$,
there are $\boldsymbol{Y} \in D_{a^*}, \boldsymbol{d} \in D_{\mathrm{sa}}, \boldsymbol{g} \in D_{ba}$, and $\boldsymbol{z} \in D_b$ such that:
$\boldsymbol{C} = {}_{\mathrm{sa}}(\boldsymbol{d}), \boldsymbol{Y} = \dot\kappa_a(\boldsymbol{d}(\boldsymbol{w})), \boldsymbol{F} = \dot\kappa_{ba}(\boldsymbol{g}), \boldsymbol{X} = \dot\kappa_b(\boldsymbol{z})$, and $\boldsymbol{Y} = \dot\kappa_a(\boldsymbol{g}(\boldsymbol{z}))$
Ψ-/Φ-Lemmas

iff for all $\boldsymbol{F} \in D_{(ba)^*}$, $\boldsymbol{X} \in D_{b^*}, \boldsymbol{w} \in D_{\mathrm{s}}, \boldsymbol{h} \in D_{\mathrm{s}(ba)}$ and $\boldsymbol{c} \in D_{\mathrm{s}b}$ such that:
$\llbracket(\lambda i.\ \beta)\rrbracket^{F,g} = \boldsymbol{h}, \boldsymbol{F} = \dot\kappa_{ba}(\boldsymbol{h}(\boldsymbol{w})), \llbracket(\lambda i.\ \gamma)\rrbracket^{F,g} = \boldsymbol{c}$, and $\boldsymbol{X} = \dot\kappa_b(\boldsymbol{c}(\boldsymbol{w}))$
there are $\boldsymbol{Y} \in D_{a^*}, \boldsymbol{d} \in D_{\mathrm{sa}}, \boldsymbol{g} \in D_{ba}$, and $\boldsymbol{z} \in D_b$ such that:
$\boldsymbol{C} = {}_{\mathrm{sa}}(\boldsymbol{d}), \boldsymbol{Y} = \dot\kappa_a(\boldsymbol{d}(\boldsymbol{w})), \boldsymbol{F} = \dot\kappa_{ba}(\boldsymbol{g})$,
$\boldsymbol{X} = \dot\kappa_b(\boldsymbol{z})$, and $\boldsymbol{Y} = \dot\kappa_a(\boldsymbol{g}(\boldsymbol{z}))$ $\dot\kappa_{\mathrm{s}b}$ injective

iff for all $\boldsymbol{w} \in D_{\mathrm{s}}$ there are $\boldsymbol{d} \in D_{\mathrm{sa}}, \boldsymbol{g} \in D_{ba}$, and $\boldsymbol{z} \in D_b$ such that:
$\boldsymbol{C} = {}_{\mathrm{sa}}(\boldsymbol{d}), \dot\kappa_{ba}(\llbracket(\lambda i.\ \beta)\rrbracket^{F,g}(\boldsymbol{w})) = \dot\kappa_{ba}(\boldsymbol{g})$,
$\dot\kappa_b(\llbracket(\lambda i.\ \gamma)\rrbracket^{F,g}(\boldsymbol{w})) = \dot\kappa_b(\boldsymbol{z})$, and $\dot\kappa_a(\boldsymbol{d}(\boldsymbol{w})) = \dot\kappa_a(\boldsymbol{g}(\boldsymbol{z}))$ identity

iff for all $\boldsymbol{w} \in D_{\mathrm{s}}$ there are $\boldsymbol{d} \in D_{\mathrm{sa}}, \boldsymbol{g} \in D_{ba}$, and $\boldsymbol{z} \in D_b$ such that:
$\dot\kappa^{-1}_{\mathrm{sa}}(\boldsymbol{C}) = \boldsymbol{d}, \llbracket(\lambda i.\ \beta)\rrbracket^{F,g}(\boldsymbol{w}) = \boldsymbol{g}$,
$\llbracket(\lambda i.\ \gamma)\rrbracket^{F,g}(\boldsymbol{w}) = \boldsymbol{z}$, and $\boldsymbol{d}(\boldsymbol{w}) = \boldsymbol{g}(\boldsymbol{z})$ $\dot\kappa_{\mathrm{sa}}, \dot\kappa_{ba}, \dot\kappa_b, \dot\kappa_a$ injective

iff for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $\dot\kappa^{-1}_{\mathrm{sa}}(\boldsymbol{C})(\boldsymbol{w}) = \llbracket(\lambda i.\ \beta)\rrbracket^{F,g}(\boldsymbol{w})(\llbracket(\lambda i.\ \gamma)\rrbracket^{F,g}(\boldsymbol{w}))$ identity

iff for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $\dot\kappa^{-1}_{\mathrm{sa}}(\boldsymbol{C})(\boldsymbol{w}) = \llbracket\beta\rrbracket^{F,g[^i/\boldsymbol{w}]}(\llbracket\gamma\rrbracket^{F,g[^i/\boldsymbol{w}]})$ IL^*-interpretation

iff for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $\dot\kappa^{-1}_{\mathrm{sa}}(\boldsymbol{C})(\boldsymbol{w}) = \llbracket\beta(\gamma)\rrbracket^{F,g[^i/\boldsymbol{w}]}$ IL^*-interpretation

iff for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $\dot\kappa^{-1}_{\mathrm{sa}}(\boldsymbol{C})(\boldsymbol{w}) = \llbracket(\lambda i.\ \beta(\gamma))\rrbracket^{F,g}(\boldsymbol{w})$ IL^*-interpretation

iff $\dot\kappa^{-1}_{\mathrm{sa}}(\boldsymbol{C}) = \llbracket(\lambda i.\ \beta(\gamma))\rrbracket^{F,g}$ extensionality

iff $\boldsymbol{C} = \dot\kappa_{\mathrm{sa}}(\llbracket(\lambda i.\ \beta(\gamma))\rrbracket^{F,g})$ def. $\dot\kappa^{-1}_{\mathrm{sa}}$

Given (!!), we thus have:

$\llbracket\overline{\alpha}\rrbracket^{F^*,g^*} = \dot\sqcap_{(\mathrm{sa})^*}(/\dot\kappa_{\mathrm{sa}}(\llbracket(\lambda i.\ \alpha)\rrbracket^{F,g})/)$,

from which we may – as above – conclude that:
$[\![\overline{\alpha}]\!]^{F^*,g^*} = \dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ \alpha)]\!]^{F,g})$.

– If $\alpha = (\lambda x.\ \beta)$, where $x \in Var_b$, $\beta \in IL_b^*$, and $a = (bc) \in \mathbf{FT}$, then $\overline{\alpha}$ is again of the form $\sqcap_{(\mathrm{s}a)^*}(\delta)$, where $\delta \in IL^*_{(\mathrm{s}a)^*\mathrm{t}}$. As before, we first show that:

(!!!) $[\![\delta]\!]^{F^*,g^*} = /\dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ (\lambda x.\ \beta))]\!]^{F,g})/$.

Indeed, for any $\boldsymbol{G} \in D_{(\mathrm{s}a)^*}$, we have:

$[\![\delta]\!]^{F^*,g^*}(\boldsymbol{G}) = 1$

iff for all $\boldsymbol{X} \in D_{b^*}$ such that $[\![\Gamma_b]\!](\boldsymbol{X}) = 1$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$ there are $\boldsymbol{F} \in D_{a^*}$ and $\boldsymbol{Y} \in D_{c^*}$ such that:
$[\![\Psi_a]\!](\boldsymbol{G})(\boldsymbol{F})(\boldsymbol{w}) = [\![\Psi_c]\!]([\![\overline{\beta}]\!]^{F^*,g^*[\tilde{x}/\boldsymbol{X}]})(\boldsymbol{Y})(\boldsymbol{w}) =$
$[\![\Phi_a]\!](\boldsymbol{F})(\boldsymbol{X})(\boldsymbol{Y}) = 1$ — IL^*-interpretation

iff for all $\boldsymbol{X} \in rge(\dot{\kappa}_b)$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$ there are $\boldsymbol{F} \in D_{a^*}$ and $\boldsymbol{Y} \in D_{c^*}$ such that, for some $\boldsymbol{g} \in D_{\mathrm{s}a}, \boldsymbol{f} \in D_a$, $\boldsymbol{c} \in D_{\mathrm{s}c}$, and $\boldsymbol{x} \in D_b$:
$\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}a}(\boldsymbol{g}), \boldsymbol{F} = \dot{\kappa}_a(\boldsymbol{g}(\boldsymbol{w}))$,
$[\![\overline{\beta}]\!]^{F^*,g^*[\tilde{x}/\boldsymbol{X}]} = \dot{\kappa}_{\mathrm{s}c}(\boldsymbol{c})$, $\boldsymbol{Y} = \dot{\kappa}_c(\boldsymbol{c}(\boldsymbol{w}))$,
$\boldsymbol{F} = \dot{\kappa}_a(\boldsymbol{f}), \boldsymbol{X} = \dot{\kappa}_b(\boldsymbol{x})$, and $\boldsymbol{Y} = \dot{\kappa}_c(\boldsymbol{f}(\boldsymbol{x}))$ — Ψ- and Φ-Lemmas

iff for all $\boldsymbol{X} \in rge(\dot{\kappa}_b)$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$ there are $\boldsymbol{F} \in D_{a^*}$ and $\boldsymbol{Y} \in D_{c^*}$ such that: for some $\boldsymbol{g} \in D_{\mathrm{s}a}, \boldsymbol{f} \in D_a$, $\boldsymbol{c} \in D_{\mathrm{s}b}$, and $\boldsymbol{x} \in D_b$:
$\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}a}(\boldsymbol{g}), \boldsymbol{F} = \dot{\kappa}_a(\boldsymbol{f}), \boldsymbol{g}(\boldsymbol{w}) = \boldsymbol{f}$,
$[\![\overline{\beta}]\!]^{F^*,g^*[\tilde{x}/\boldsymbol{X}]} = \dot{\kappa}_{\mathrm{s}c}(\boldsymbol{c})$, $\boldsymbol{Y} = \dot{\kappa}_c(\boldsymbol{y}), \boldsymbol{c}(\boldsymbol{w}) = \boldsymbol{y}$,
$\boldsymbol{X} = \dot{\kappa}_b(\boldsymbol{x})$, and $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}$ — injectivity of $\dot{\kappa}_c$ and $\dot{\kappa}_a$

iff for all $\boldsymbol{x} \in D_b$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$ there are $\boldsymbol{g} \in D_{\mathrm{s}a}, \boldsymbol{f} \in D_a$ and $\boldsymbol{c} \in D_{\mathrm{s}b}$ such that:
$\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}a}(\boldsymbol{g}), \boldsymbol{g}(\boldsymbol{w}) = \boldsymbol{f}, [\![\overline{\beta}]\!]^{F^*,g^*[\tilde{x}/\dot{\kappa}_b(\boldsymbol{x})]} = \dot{\kappa}_{\mathrm{s}c}(\boldsymbol{c})$, and $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{c}(\boldsymbol{w})$ — identity

iff for all $\boldsymbol{x} \in D_b$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$ there are $\boldsymbol{g} \in D_{\mathrm{s}a}, \boldsymbol{f} \in D_a$ and $\boldsymbol{c} \in D_{\mathrm{s}b}$ such that:
$\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}a}(\boldsymbol{g}), \boldsymbol{g}(\boldsymbol{w}) = \boldsymbol{f}, [\![\overline{\beta}]\!]^{F^*,g[x/\boldsymbol{x}]^*} = \dot{\kappa}_{\mathrm{s}c}(\boldsymbol{c})$, and $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{c}(\boldsymbol{w})$ — n)

iff for all $\boldsymbol{x} \in D_b$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$ there are $\boldsymbol{g} \in D_{\mathrm{s}a}, \boldsymbol{f} \in D_a$ and $\boldsymbol{c} \in D_{\mathrm{s}b}$ such that:
$\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}a}(\boldsymbol{g}), \boldsymbol{g}(\boldsymbol{w}) = \boldsymbol{f}, \dot{\kappa}_{\mathrm{s}c}([\![(\lambda i.\ \beta)]\!]^{F,g[x/\boldsymbol{x}]}) = \dot{\kappa}_{\mathrm{s}c}(\boldsymbol{c})$, and $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{c}(\boldsymbol{w})$ — induction

iff for all $\boldsymbol{x} \in D_b$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$ there are $\boldsymbol{g} \in D_{\mathrm{s}a}, \boldsymbol{f} \in D_a$ and $\boldsymbol{c} \in D_{\mathrm{s}b}$ such that:
$\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}a}(\boldsymbol{g}), \boldsymbol{g}(\boldsymbol{w}) = \boldsymbol{f}, [\![(\lambda i.\ \beta)]\!]^{F,g[x/\boldsymbol{x}]} = \boldsymbol{c}$, and $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{c}(\boldsymbol{w})$ — injectivity of $\dot{\kappa}_{\mathrm{s}c}$

iff for all $\boldsymbol{x} \in D_b$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$ there is a $\boldsymbol{g} \in D_{\mathrm{s}a}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}a}(\boldsymbol{g})$ and $\boldsymbol{g}(\boldsymbol{w})(\boldsymbol{x}) = [\![(\lambda i.\ \beta)]\!]^{F,g[x/\boldsymbol{x}]}(\boldsymbol{w})$ — identity

iff for all $\boldsymbol{x} \in D_b$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$ there is a $\boldsymbol{g} \in D_{\mathrm{s}a}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{s}a}(\boldsymbol{g})$ and $\boldsymbol{g}(\boldsymbol{w})(\boldsymbol{x}) = [\![\beta]\!]^{F,g[x/\boldsymbol{x}][i/\boldsymbol{w}]}$ — IL^*-interpretation

iff for all $\boldsymbol{x} \in D_b$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$ there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$
and $\boldsymbol{g}(\boldsymbol{w})(\boldsymbol{x}) = [\![\beta]\!]^{F,g[^i/_{\boldsymbol{w}}][^x/_{\boldsymbol{x}}]}$ since $i \neq x$, $g[^x/_{\boldsymbol{x}}][^i/_{\boldsymbol{w}}] = g[^i/_{\boldsymbol{w}}][^x/_{\boldsymbol{x}}]$
iff for all $\boldsymbol{x} \in D_b$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$ there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$
and $\boldsymbol{g}(\boldsymbol{w})(\boldsymbol{x}) = [\![(\lambda x.\ \beta)]\!]^{F,g[^i/_{\boldsymbol{w}}]}(\boldsymbol{x})$ IL^*-interpretation
iff for all $\boldsymbol{x} \in D_b$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$ there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\dot{\kappa}_{\mathrm{sa}}^{-1}(\boldsymbol{G}) = \boldsymbol{g}$
and $\boldsymbol{g}(\boldsymbol{w})(\boldsymbol{x}) = [\![(\lambda x.\ \beta)]\!]^{F,g[^i/_{\boldsymbol{w}}]}(\boldsymbol{x})$ injectivity of $\dot{\kappa}_{\mathrm{sa}}$
iff for all $\boldsymbol{x} \in D_b$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$:
$\dot{\kappa}_{\mathrm{sa}}^{-1}(\boldsymbol{G})(\boldsymbol{w})(\boldsymbol{x}) = [\![(\lambda x.\ \beta)]\!]^{F,g[^i/_{\boldsymbol{w}}]}(\boldsymbol{w})(\boldsymbol{x})$ identity
iff for all $\boldsymbol{x} \in D_b$ and all $\boldsymbol{w} \in D_{\mathrm{s}}$:
$\dot{\kappa}_{\mathrm{sa}}^{-1}(\boldsymbol{G})(\boldsymbol{w})(\boldsymbol{x}) = [\![(\lambda i.\ (\lambda x.\ \beta))]\!]^{F,g}(\boldsymbol{w})(\boldsymbol{x})$ IL^*-interpretation
iff $\dot{\kappa}_{\mathrm{sa}}^{-1}(\boldsymbol{G}) = [\![(\lambda i.\ (\lambda x.\ \beta))]\!]^{F,g}$ extensionality
iff $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ (\lambda x.\ \beta))]\!]^{F,g})$ def. $\dot{\kappa}_{\mathrm{sa}}$

...which establishes (!!!). As before, we may now apply **s)** and conclude that:

$$[\![\overline{\alpha}]\!]^{F^*,g^*} = \dot{\sqcap}_{\mathrm{sa}}(/\dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ \alpha)]\!]^{F,g})/) = \dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ \alpha)]\!]^{F,g}).$$

– If $\alpha = \beta(i)$, where $\beta \in IL^*_{\mathrm{sa}}$, $\overline{\alpha}$ is again of the form $\sqcap_{(\mathrm{sa})^*}(\delta)$, where $\delta \in IL^*_{(\mathrm{sa})^*\mathrm{t}}$; and again we can show that:

(!4) $[\![\delta]\!]^{F^*,g^*} = /\dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ \beta(\gamma))]\!]^{F,g})/.$

For we have, for any $\boldsymbol{G} \in D_{(\mathrm{sa})^*}$:

$[\![\delta]\!]^{F^*,g^*}(\boldsymbol{G}) = 1$
iff for all $\boldsymbol{w} \in D_{\mathrm{s}}$ there are $\boldsymbol{H} \in D_{(\mathrm{sa})^*}$ and $\boldsymbol{X} \in D_{a^*}$ such that:
$[\![\Psi_{\mathrm{sa}}]\!]([\![\overline{\beta}]\!]^{F^*,g^*})(\boldsymbol{H})(\boldsymbol{w}) = [\![\Psi_a]\!](\boldsymbol{H})(\boldsymbol{X})(\boldsymbol{w}) = [\![\Psi_a]\!](\boldsymbol{G})(\boldsymbol{X})(\boldsymbol{w}) = 1$
IL^*-interpretation
iff for all $\boldsymbol{w} \in D_{\mathrm{s}}$ there are $\boldsymbol{H} \in D_{(\mathrm{sa})^*}$, $\boldsymbol{X} \in D_{a^*}$,
$\boldsymbol{b} \in D_{\mathrm{s(sa)}}$, $\boldsymbol{g}, \boldsymbol{h} \in D_{\mathrm{s}a}$ such that:
$[\![\overline{\beta}]\!]^{F^*,g^*} = \dot{\kappa}_{\mathrm{s(sa)}}(\boldsymbol{b})$, $\boldsymbol{H} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{b}(\boldsymbol{w}))$,
$\boldsymbol{H} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{h})$, $\boldsymbol{X} = \dot{\kappa}_a(\boldsymbol{h}(\boldsymbol{w}))$,
$\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$, and $\boldsymbol{X} = \dot{\kappa}_a(\boldsymbol{g}(\boldsymbol{w}))$ Ψ-Lemma
iff for all $\boldsymbol{w} \in D_{\mathrm{s}}$ there are $\boldsymbol{b} \in D_{\mathrm{s(sa)}}$, $\boldsymbol{g}, \boldsymbol{h} \in D_{\mathrm{sa}}$ such that:
$[\![\overline{\beta}]\!]^{F^*,g^*} = \dot{\kappa}_{\mathrm{s(sa)}}(\boldsymbol{b})$, $\dot{\kappa}_{\mathrm{sa}}(\boldsymbol{b}(\boldsymbol{w})) = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{h})$, $\dot{\kappa}_a(\boldsymbol{h}(\boldsymbol{w})) = \dot{\kappa}_a(\boldsymbol{g}(\boldsymbol{w}))$,
and $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$ identity
iff there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$ and
for all $\boldsymbol{w} \in D_{\mathrm{s}}$ there are $\boldsymbol{b} \in D_{\mathrm{s(sa)}}$ and $\boldsymbol{h} \in D_{\mathrm{s}a}$ such that:
$[\![\overline{\beta}]\!]^{F^*,g^*} = \dot{\kappa}_{\mathrm{s(sa)}}(\boldsymbol{b})$, $\boldsymbol{b}(\boldsymbol{w}) = \boldsymbol{h}$, and $\boldsymbol{h}(\boldsymbol{w}) = \boldsymbol{g}(\boldsymbol{w})$ injectivity of $\dot{\kappa}_{\mathrm{sa}}, \dot{\kappa}_a$
iff there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$ and
for all $\boldsymbol{w} \in D_{\mathrm{s}}$ there is a $\boldsymbol{b} \in D_{\mathrm{s(sa)}}$ such that:
$[\![\overline{\beta}]\!]^{F^*,g^*} = \dot{\kappa}_{\mathrm{s(sa)}}(\boldsymbol{b})$ and $\boldsymbol{b}(\boldsymbol{w})(\boldsymbol{w}) = \boldsymbol{g}(\boldsymbol{w})$ identity
iff there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$ and
for all $\boldsymbol{w} \in D_{\mathrm{s}}$ there is a $\boldsymbol{b} \in D_{\mathrm{s(sa)}}$ such that:
$\dot{\kappa}_{\mathrm{s(sa)}}([\![(\lambda i.\ \beta)]\!]^{F,g}) = \dot{\kappa}_{\mathrm{s(sa)}}(\boldsymbol{b})$ and $\boldsymbol{b}(\boldsymbol{w})(\boldsymbol{w}) = \boldsymbol{g}(\boldsymbol{w})$ induction

iff there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$ and
for all $\boldsymbol{w} \in D_{\mathrm{s}}$ there is a $\boldsymbol{b} \in D_{\mathrm{s(sa)}}$ such that:
$[\![(\lambda i.\ \beta)]\!]^{F,g} = \boldsymbol{b}$ and $\boldsymbol{b}(\boldsymbol{w})(\boldsymbol{w}) = \boldsymbol{g}(\boldsymbol{w})$ — injectivity of $\dot{\kappa}_{\mathrm{s}(sa)}$
iff there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$
and for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $[\![(\lambda i.\ \beta)]\!]^{F,g}(\boldsymbol{w})(\boldsymbol{w}) = \boldsymbol{g}(\boldsymbol{w})$ — identity
iff there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$
and for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $[\![\beta]\!]^{F,g[^i/_{\boldsymbol{w}}]}(\boldsymbol{w}) = \boldsymbol{g}(\boldsymbol{w})$ — IL^*-interpretation
iff there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$
and for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $[\![\beta]\!]^{F,g[^i/_{\boldsymbol{w}}]}(g[^i/_{\boldsymbol{w}}](i)) = \boldsymbol{g}(\boldsymbol{w})$ — def. $g[^i/_{\boldsymbol{w}}]$
iff there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$
and for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $[\![\beta]\!]^{F,g[^i/_{\boldsymbol{w}}]}([\![i]\!]^{F,g[^i/_{\boldsymbol{w}}]}) = \boldsymbol{g}(\boldsymbol{w})$ — IL^*-interpretation
iff there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$
and for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $[\![\beta(i)]\!]^{F,g[^i/_{\boldsymbol{w}}]} = \boldsymbol{g}(\boldsymbol{w})$ — IL^*-interpretation
iff there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$
and for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $[\![\lambda i.\ \beta(i))]\!]^{F,g}(\boldsymbol{w}) = \boldsymbol{g}(\boldsymbol{w})$ — IL^*-interpretation
iff there is a $\boldsymbol{g} \in D_{\mathrm{sa}}$ such that $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{g})$ and $[\![\lambda i.\ \beta(i))]\!]^{F,g} = \boldsymbol{g}$
— extensionality
iff $\boldsymbol{G} = \dot{\kappa}_{\mathrm{sa}}([\![\lambda i.\ \beta(i))]\!]^{F,g})$ — identity

…and we may again conclude that:

$$[\![\overline{\alpha}]\!]^{F^*,g^*} = [\![\sqcap_{\mathrm{sa}}]\!](/\dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ \alpha)]\!]^{F,g})/) = \dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ \alpha)]\!]^{F,g}).$$

– If $\alpha = (\lambda i.\beta)$, where $\beta \in IL^*_b$, then $\overline{\alpha}$ is of the form $\sqcap(\delta)$ again, where $\delta \in IL^*_{(\mathrm{sa})^*\mathrm{t}}$; and once more we can show that:

(!5) $[\![\delta]\!]^{F^*,g^*} = /\dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ (\lambda i.\beta))]\!]^{F,g})/.$

For we have, for any $\boldsymbol{C} \in D_{(\mathrm{sa})^*}$:

$[\![(\lambda C^{(sa)^*}.\ \Box\, \Psi_a(C)(\overline{\beta}))]\!]^{M^*,g^*}(\boldsymbol{C}) = 1$
iff for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $[\![\Psi_a]\!](\boldsymbol{C})([\![\overline{\beta}]\!]^{M^*,g^*})(\boldsymbol{w}) = 1$ — IL^*-interpretation
iff for all $\boldsymbol{w} \in D_{\mathrm{s}}$ there is a $\boldsymbol{c} \in D_{\mathrm{sa}}$ such that:
$\boldsymbol{C} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{c})$ and $[\![\overline{\beta}]\!]^{M^*,g^*} = \dot{\kappa}_a(\boldsymbol{c}(\boldsymbol{w}))$ — Ψ-Lemma
iff for all $\boldsymbol{w} \in D_{\mathrm{s}}$ there is a $\boldsymbol{c} \in D_{\mathrm{sa}}$ such that:
$\boldsymbol{C} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{c})$ and $\dot{\kappa}_a([\![(\lambda i.\ \beta)]\!]^{M,g}) = \dot{\kappa}_a(\boldsymbol{c}(\boldsymbol{w}))$ — induction
iff for all $\boldsymbol{w} \in D_{\mathrm{s}}$ there is a $\boldsymbol{c} \in D_{\mathrm{sa}}$ such that:
$\dot{\kappa}^{-1}_{\mathrm{sa}}(\boldsymbol{C}) = \boldsymbol{c}$ and $[\![(\lambda i.\ \beta)]\!]^{M,g} = \boldsymbol{c}(\boldsymbol{w})$ — injectivity of $\dot{\kappa}_{\mathrm{sa}}$ and $\dot{\kappa}_a$
iff there is a $\boldsymbol{c} \in D_{\mathrm{sa}}$ such that $\boldsymbol{C} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{c})$ and
for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $[\![(\lambda i.\ \beta)]\!]^{M,g} = \boldsymbol{c}(\boldsymbol{w})$ — injectivity of $\dot{\kappa}_a$
iff there is a $\boldsymbol{c} \in D_{\mathrm{sa}}$ such that $\boldsymbol{C} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{c})$ and
for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $[\![(\lambda i.\ \beta)]\!]^{M,g[^i/_{\boldsymbol{w}}]} = \boldsymbol{c}(\boldsymbol{w})$ — $i \notin Fr((\lambda i.\ \beta))$
iff there is a $\boldsymbol{c} \in D_{\mathrm{sa}}$ such that $\boldsymbol{C} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{c})$ and
for all $\boldsymbol{w} \in D_{\mathrm{s}}$: $[\![(\lambda i.\ (\lambda i.\ \beta))]\!]^{M,g}(\boldsymbol{w}) = \boldsymbol{c}(\boldsymbol{w})$ — IL^*-interpretation
iff there is a $\boldsymbol{c} \in D_{\mathrm{sa}}$ such that $\boldsymbol{C} = \dot{\kappa}_{\mathrm{sa}}(\boldsymbol{c})$ and
$[\![(\lambda i.\ (\lambda i.\ \beta))]\!]^{M,g} = \boldsymbol{c}$ — extensionality
iff $\boldsymbol{C} = \dot{\kappa}_{\mathrm{sa}}([\![(\lambda i.\ (\lambda i.\ \beta))]\!]^{M,g})$ — identity

Now (!5) follows as before.

References

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